

Derived Witt-Dévissage Formalism

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Abstract: *In this article we establish some formalism of Derived Witt-Dévissage theory for resolving subcategories of abelian categories. Results directly apply to noetherian schemes.*

1 Introduction

This is another one of a series of articles ([MS1, MS2, MS3]) dedicated to study of derived Witt groups. Our interest in derived Witt groups emanates from our interest in Chow-Witt groups, as obstruction groups for projective modules. We would refrain from repetition of the introductory comments and the background discussion provided in ([MS1]).

In ([MS1]), we proved dévissage theorems for Cohen-Macaulay affine schemes, extending the corresponding theorem of Balmer and Walter ([BW]) on regular schemes. In this paper, we prove analogous dévissage theorems for noetherian schemes X , with the hypothesis that any coherent sheaf on X is a quotient of a locally free sheaf on X . In particular, we remove the Cohen-Macaulay condition in ([MS1]). In the process of doing so, it became clear that the arguments could be made formal enough, so that they work for resolving subcategories of abelian categories. The concept of resolving subcategories dates back to the paper of Auslander and Bridger ([AB]). More recently, there has been considerable amount of activities (*e.g.* [T1, T2]) on resolving subcategories of the category $Mod(A)$ of finitely generated modules over noetherian commutative rings A . Much of it is directed toward the classification of such resolving subcategories under various conditions, which is encompassed by similar attempts to classify variety of types of subcategories of the module categories. The work of Benson, Iyengar and Krause (*e.g.* [BIK]) would be one of the stimulus, where they consider subcategories of group algebras. To the best of our knowledge, very little literature is available on resolving subcategories of abelian categories. We give a version of the dévissage theorem for resolving subcategories of abelian categories,

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which directly applies to noetherian schemes. From our perspective, the results on Witt groups of noetherian schemes constitute the essence of this article. However, the formal version of the results on resolving subcategories unifies the theory and has its significance by its own rights. In this introduction, we will state the versions of the results for resolving subcategories, as follows.

Suppose \mathcal{V} is a resolving subcategory (see definition 3.11) of an abelian category \mathcal{C} . Let ω be an object in \mathcal{C} with an injective resolution. For objects M in \mathcal{C} , denote $M^* := \text{Hom}(M, \omega)$. Assume \mathcal{V} inherits a duality structure from $*$ and \mathcal{V} is totally ω -reflexive (see definition 7.1). The following is a list of a few notations and facts.

1. Let $\mathcal{B}(\mathcal{V}, \omega)$ denote the full subcategory of objects in \mathcal{C} with finite \mathcal{V} -dimension. Assume $d := \max\{\dim_{\mathcal{V}} M : M \in \mathcal{B}(\mathcal{V}, \omega)\} < \infty$.
2. Let $\mathcal{A}(\omega) := \{M \in \mathcal{B}(\mathcal{V}, \omega) : \text{Ext}^i(M, \omega) = 0 \ \forall i < d\}$. It follows that $M \mapsto M^\vee := \text{Ext}^d(M, \omega)$ is a duality on $\mathcal{A}(\omega)$.
3. $D^b(\mathcal{V})$ would denote the bounded derived category of complexes of objects in \mathcal{V} and $D_{\mathcal{A}(\omega)}^b(\mathcal{V})$ would be the subcategory of complexes with homologies in $\mathcal{A}(\omega)$.
4. It is a standard fact (see 3.13) that there is a canonical functor $\zeta : \mathcal{B}(\mathcal{V}, \omega) \longrightarrow D^b(\mathcal{V})$, given by (choices of) \mathcal{V} -resolutions.
5. The restriction of $\zeta : \mathcal{A}(\omega) \longrightarrow D_{\mathcal{A}(\omega)}^b(\mathcal{V})$ is a duality preserving functor.

First, we prove isomorphisms of Witt groups, as follows.

$$W(\mathcal{A}(\omega),^\vee, \pm\tilde{\omega}) \xrightarrow{\sim} W(D_{\mathcal{A}(\omega)}^b(\mathcal{A}(\omega),^\vee, \pm\tilde{\omega})) \xrightarrow{\sim} W(D^b(\mathcal{A}(\omega),^\vee, \pm\tilde{\omega})).$$

It follows from a theorem of Balmer ([B3]) that the first and the last groups are isomorphic. We further prove that the functor ζ induces isomorphisms of Witt groups

$$W_{St}^+(\mathcal{A}(\omega)) \xrightarrow{\sim} W^d(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, \varpi), \quad W_{St}^-(\mathcal{A}(\omega)) \xrightarrow{\sim} W^{d-2}(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, -\varpi)$$

where subscript "St" corresponds to "standard" sign convention of the duality. Also for $n = d - 1, d - 3$, we have $W^n(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, \pm\varpi) = 0$. By 4-periodicity all the shifted Witt groups are determined.

These results can be applied to noetherian schemes X , with $\mathcal{C} = \text{Coh}(X)$ and \mathcal{V} as the subcategory of locally free sheaves on X , provided \mathcal{V} is a resolving subcategory. This will be the case for a wide variety of schemes X (see [H]), including those that have an ample invertible sheaf. In these applications, we assume $d := \dim X = \max\{\text{depth}(\mathcal{O}_{X,x}) : x \text{ is a closed point}\}$ (see remark 6.5). As a consequence, the following decomposition theorem follows.

Theorem 1.1. *Suppose X is a noetherian scheme, with $\dim X = d$, as in (2.1) and $X^{(d)}$ will denote the set of all closed points of codimension d in X . We assume $d =$*

$\max\{\text{depth}(\mathcal{O}_{X,x}) : x \text{ is a closed point}\} = \dim X$. Then, the homomorphisms

$$W^d(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, \varpi) \xrightarrow{\sim} \bigoplus_{x \in X^{(d)}} W^d(D_{\mathcal{A}(\mathcal{O}_{X,x})}^b(\mathcal{O}_{X,x}), *, 1, \varpi) \quad \text{and}$$

$$W^{d-2}(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, -\varpi) \xrightarrow{\sim} \bigoplus_{x \in X^{(d)}} W^{d-2}(D_{\mathcal{A}(\mathcal{O}_{X,x})}^b(\mathcal{O}_{X,x}), *, 1, -\varpi)$$

are isomorphisms.

When X is regular, this is a theorem of Balmer and Walter ([BW]). In this case, first isomorphism follows from such a decomposition of the corresponding categories and the second isomorphism would have zeros on both sides.

Before we conclude this introduction, we comment on the sense of direction of this series of articles ([MS1, MS2, MS3]). For noetherian schemes X as above, we will give a Gersten-Witt like complex of the "relative" Witt groups $W^p(\mathcal{D}^p(X), \mathcal{D}^{p+1}(X))$, where $\mathcal{D}^p(X)$ denotes the subcategory of the finite derived category $D^b(X)$, of complexes \mathcal{E}_\bullet with finite locally free dimension homologies $\mathcal{H}_i(\mathcal{E}_\bullet)$ and $\text{codim}(\mathcal{H}_i(\mathcal{E}_\bullet)) \geq p$. The "relative" Witt groups are defined by forming a group of isometry classes of S -spaces, as in ([B2]). These results will appear subsequently.

We close this introduction, with a few comments on the lay out of this article. As would be expected, the results on Witt groups of noetherian schemes would follow from that of the resolving subcategories. However, we made a choice to give complete proofs for the former (§ 5, 6) and the proofs of the results on resolving subcategories (§ 7) would follow similarly. Throughout in § 5, 6, we remind the readers that our proofs are formal enough to apply to § 7. We also give a proof of the existence of the functor $\zeta : \mathcal{B}(X) \rightarrow D^b(\mathcal{V}(X))$ in § 3. While this is a standard fact among the experts, we were unable to find references that would be of any help for the readership of this article. While the main isomorphism theorem was dealt with in § 5, we summarize our results on noetherian schemes in § 6. In § 7, we state our results on resolving subcategories.

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2 Some Notations

First, we borrow some notations from ([MS1]).

Notations 2.1. What follows would be our standard set up, throughout this article.

Throughout this paper, $X := (X, \mathcal{O}_X)$ will denote a noetherian scheme, with $\dim X = d$. We also assume 2 is invertible in \mathcal{O}_X . Unless stated otherwise, we assume that every

coherent sheaf on X is quotient of a locally free sheaf on X . This hypothesis is satisfied in the following two cases: (1) when X is integral, locally factorial and separated (see [H, Ex. 6.8]) and (2) when X has an ample invertible sheaf. To avoid technicalities (see remark 6.5), we assume

$$d := \dim X = \max\{\text{depth}(\mathcal{O}_{X,x}) : x \text{ is a closed point}\}.$$

This ensures that $\mathcal{A}(X)$ has nonzero symmetric forms. We set up further notations:

1. $\text{Coh}(X)$ will denote the category of coherent \mathcal{O}_X -modules and $\mathcal{V}(X)$ will denote the full subcategory of locally free sheaves on X .
2. The full subcategory of objects \mathcal{F} in $\text{Coh}(X)$, so that \mathcal{F} has finite resolution by locally free sheaves, will be denoted by $\mathcal{B} := \mathcal{B}(X)$.
3. Also, let $\mathcal{A} := \mathcal{A}(X) \subseteq \mathcal{B}(X)$ be the full subcategory of objects $\mathcal{F} \in \mathcal{B}(X)$ such that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i < d$. A coherent sheaf $\mathcal{F} \in \mathcal{A}(X)$ if and only if $\text{codim}(\text{Supp}(\mathcal{F})) = d$ and \mathcal{F}_x has finite projective dimension for all closed points $x \in X$. *(For our subsequent discussions, these two notations \mathcal{B}, \mathcal{A} will be of some importance. In the absence of more standard notations or abbreviations of what they are, we use these two notations \mathcal{B}, \mathcal{A} . However, both \mathcal{B} and \mathcal{A} are abelian subcategories of $\text{Coh}(X)$, which justifies at least one of these two notations.)*
4. For any exact category \mathcal{C} , $Ch^b(\mathcal{C}), D^b(\mathcal{C})$ will denote the category of bounded chain complexes, and respectively, derived category. If \mathcal{C} is a subcategory in an ambient abelian category \mathcal{C}' and \mathcal{H} is a subcategory of \mathcal{C}' , then $Ch_{\mathcal{H}}^b(\mathcal{C})$ denotes the full subcategory of $Ch^b(\mathcal{C})$ consisting of complexes with homologies in \mathcal{H} . The derived category of $Ch_{\mathcal{H}}^b(\mathcal{C})$ will be denoted by $D_{\mathcal{H}}^b(\mathcal{C})$, which is obtained by inverting quasi-isomorphisms in $Ch_{\mathcal{H}}^b(\mathcal{C})$. However, $D_{\mathcal{H}}^b(\mathcal{C})$ can also be viewed as the full subcategory of $D^b(\mathcal{C})$ consisting of objects from $Ch_{\mathcal{H}}^b(\mathcal{C})$. Also, $K^b(\mathcal{C}), K_{\mathcal{H}}^b(\mathcal{C})$ would denote the corresponding homotopy categories. For other similar notations, readers are referred to ([W]).
5. Denote objects $\mathcal{E}_{\bullet} := (\mathcal{E}_{\bullet}, \partial_{\bullet})$ in $Ch^b(\text{Coh}(X))$ as:

$$\cdots 0 \longrightarrow \mathcal{E}_m \xrightarrow{\partial_m} \mathcal{E}_{m-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_n \longrightarrow 0 \cdots \text{ with } m > n.$$
6. A complex \mathcal{E}_{\bullet} is said to be supported on $[m, n]$ if $\mathcal{E}_i = 0$ unless $m \geq i \geq n$.
7. For a complex \mathcal{E}_{\bullet} in $Ch^b(\mathcal{V}(X))$, $\mathcal{E}_{\bullet}^{\#}$ will denote the usual dual induced by sheaf $\mathcal{H}om(-, \mathcal{O}_X)$ and $\varpi : \mathcal{E}_{\bullet} \xrightarrow{\sim} \mathcal{E}_{\bullet}^{\#\#}$ will denote the evaluation map.

8. Let $B_r = B_r(\mathcal{E}_\bullet) := \partial_{r+1}(\mathcal{E}_{r+1}) \subseteq \mathcal{E}_r$ denote the module of r -boundaries and $\mathcal{Z}_r = \mathcal{Z}_r(\mathcal{E}_\bullet) := \ker(\partial_r) \subseteq \mathcal{E}_r$ denote the module of r -cycles.
9. The r^{th} -homology of \mathcal{E}_\bullet will be denoted by $\mathcal{H}_r := \mathcal{H}_r(\mathcal{E}_\bullet)$ and is defined by the exact sequence $0 \longrightarrow B_r(\mathcal{E}_\bullet) \longrightarrow \mathcal{Z}_r(\mathcal{E}_\bullet) \longrightarrow \mathcal{H}_r \longrightarrow 0$.

3 The functor by resolution

Let X be a noetherian scheme, as in (2.1). When $X = \text{Spec}(A)$ is affine, it is not difficult to see that there is a natural functor $\zeta : \text{Coh}(X) \longrightarrow K^{\geq 0}(\mathcal{V}(X))$, given by (*choice of*) projective resolutions of finitely generated A -modules M in $\text{Coh}(X)$. If $M \in \mathcal{B}$ has finite projective dimension then $\zeta(M) \in K^b(\mathcal{V}(X))$. When X is not affine, morphisms $f : \mathcal{F} \longrightarrow \mathcal{G}$ in $\text{Coh}(X)$ do not naturally lift to morphisms of the corresponding $\mathcal{V}(X)$ -resolutions of \mathcal{F} and \mathcal{G} .

However, even when X is not affine there are, instead functors $\zeta : \mathcal{B} \longrightarrow D^b(\mathcal{V}(X))$, (resp. $\zeta : \mathcal{B}(X) \longrightarrow D^{\geq 0}(\mathcal{V}(X))$) to the derived category, given by (*the choices*) of $\mathcal{V}(X)$ -resolutions of objects in \mathcal{B} (resp. of $\text{Coh}(X)$). While this fact is well known among the experts, there is no suitable reference, to the best of our knowledge, that is accessible to a wider community. The purpose of this section is to give a proof of this fact for the benefit of such readers. We point out in § 3.1, that one can define and prove the existence of such a functor in the context of resolving subcategories of abelian categories.

Lemma 3.1. *Suppose X is a noetherian scheme as in (2.1). Let $g : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of coherent sheaves and $\cdots \longrightarrow \mathcal{G}_r \longrightarrow \mathcal{G}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow 0$ be an exact sequence of coherent \mathcal{O}_X -modules. Then, there is a resolution \mathcal{L}_\bullet of \mathcal{F} by objects in $\mathcal{V}(X)$ and a morphism $g_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ of complexes such that the diagram commutes:*

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \mathcal{L}_r & \longrightarrow & \mathcal{L}_{r-1} & \longrightarrow & \cdots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \\
 & & \downarrow g_r & & \downarrow g_{r-1} & & \downarrow g_0 & \downarrow g \\
 \cdots & \longrightarrow & \mathcal{G}_r & \xrightarrow{\partial_r} & \mathcal{G}_{r-1} & \longrightarrow & \cdots \longrightarrow \mathcal{G}_0 \xrightarrow{\partial_0} \mathcal{G} \longrightarrow 0
 \end{array}$$

Further, if \mathcal{F} is in $\mathcal{B}(X)$ and $\mathcal{G}_k = 0 \ \forall k \gg 0$, then \mathcal{L}_\bullet can be chosen to be in $\text{Ch}^b(\mathcal{V}(X))$.

Proof. Consider the pullback diagram

$$\begin{array}{ccc}
 \Gamma_0 & \xrightarrow{p_0} & \mathcal{F} \\
 q_0 \downarrow & & \downarrow g \\
 \mathcal{G}_0 & \xrightarrow{\partial_0} & \mathcal{G}
 \end{array}$$

Since ∂_0 is surjective, so is p_0 . Since Γ_0 is coherent, by hypothesis, there is a surjective morphism $h_0 : \mathcal{L}_0 \twoheadrightarrow \Gamma_0$, where $\mathcal{L}_0 \in \mathcal{V}(X)$. Let $d_0 = p_0 h_0$ and $g_0 = q_0 h_0$. So, we have a commutative diagram of exact sequences :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}_0 & \longrightarrow & \mathcal{L}_0 & \xrightarrow{d_0} & \mathcal{F} \longrightarrow 0 \\ & & \downarrow G_0 & & \downarrow g_0 & & \downarrow g \\ 0 & \longrightarrow & B_0 & \longrightarrow & \mathcal{G}_0 & \xrightarrow{\partial_0} & \mathcal{G} \longrightarrow 0 \end{array}$$

where $B_0 = \ker(\partial_0)$, $\mathcal{Z}_0 = \ker(d_0)$, and G_0 is the restriction of g_0 . We extend the diagram as follows:

$$\begin{array}{ccccccc} \Gamma_1 & \xrightarrow{p_1} & \mathcal{Z}_0 & \longrightarrow & \mathcal{L}_0 & \xrightarrow{d_0} & \mathcal{F} \longrightarrow 0 \\ q_1 \downarrow & & \downarrow G_0 & & \downarrow g_0 & & \downarrow g \\ \mathcal{G}_1 & \xrightarrow{\partial_1} & B_0 & \longrightarrow & \mathcal{G}_0 & \xrightarrow{\partial_0} & \mathcal{G} \longrightarrow 0 \end{array}$$

Here Γ_1 is the pullback. Note p_1 is surjective. Now, the proof of the first part is complete by iteration of this process. The latter statement follows because the process terminates, in that case. \square

The following is another version of (3.1) that will be of some interest subsequently.

Lemma 3.2. *Suppose X is a noetherian scheme, as in (2.1). Let \mathcal{G}_\bullet be a complex in $Ch^+(Coh(X))$. Then there is a quasi-isomorphism $g_\bullet : \mathcal{L}_\bullet \rightarrow \mathcal{G}_\bullet$, with \mathcal{L}_\bullet in $Ch^+(\mathcal{V}(X))$. Further, if \mathcal{G}_\bullet is in $Ch^b(Coh(X))$, such that all the homologies $\mathcal{H}_i(\mathcal{G}_\bullet) \in \mathcal{B}(X)$. Then \mathcal{L}_\bullet can be chosen in $Ch^b(\mathcal{V}(X))$.*

Proof. (With § 3.1 in mind, we avoid local argument and stick to formal arguments). We write \mathcal{G}_\bullet as $\dots \rightarrow \mathcal{G}_{n+1} \rightarrow \mathcal{G}_n \xrightarrow{\partial_n} \mathcal{G}_{n-1} \rightarrow \dots$. We assume that $\mathcal{G}_i = 0 \ \forall i < 0$. Denote $B_i := \text{image}(\partial_{i+1})$ and $Z_i := \ker(\partial_i)$. First, by hypothesis, there is a surjective map $g_0 : \mathcal{L}_0 \twoheadrightarrow \mathcal{G}_0$, with $\mathcal{L}_0 \in \mathcal{V}(X)$. With $d_0 = \partial_0 g_0$ and $\mathcal{Z}_0 = \ker(d_0)$, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}_0 & \longrightarrow & \mathcal{L}_0 & \xrightarrow{d_0} & \mathcal{H}_0(\mathcal{G}_\bullet) \longrightarrow 0 \\ & & \downarrow f_0 & & \downarrow g_0 & & \parallel \\ 0 & \longrightarrow & Z_0 & \longrightarrow & \mathcal{G}_0 & \xrightarrow{\partial_0} & \mathcal{H}_0(\mathcal{G}_\bullet) \longrightarrow 0 \end{array} \quad f_0 \text{ where is the restriction of } g_0.$$

It follows f_0 is surjective. Inductively, we choose surjective maps $g_n : \mathcal{L}_n \twoheadrightarrow \mathcal{G}_n$ and differentials $d_n : \mathcal{L}_n \rightarrow \mathcal{L}_{n-1}$, such that (1) $\mathcal{L}_n \in \mathcal{V}(X)$ and (2) with $\mathcal{Z}_n := \ker(d_n)$, the restriction map $f_n : \mathcal{Z}_n \twoheadrightarrow Z_n$ of g_n to $\mathcal{Z}_n := \ker(d_n)$ is surjective. So, we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}_n & \longrightarrow & \mathcal{L}_n & \xrightarrow{d_n} & \mathcal{L}_{n-1} \\ & & \downarrow f_n & & \downarrow g_n & & \downarrow g_{n-1} \\ 0 & \longrightarrow & Z_n & \longrightarrow & \mathcal{G}_n & \xrightarrow{\partial_n} & \mathcal{G}_{n-1} \end{array}$$

Consider the pullback diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{Z}_{n+1} & \longrightarrow & \mathcal{L}_{n+1} & & \\
& & \downarrow f_{n+1} & & \downarrow \varphi & \searrow d_{n+1} & \\
0 & \longrightarrow & \mathcal{Z}_{n+1} & \longrightarrow & \Gamma_{n+1} & \xrightarrow{p} & f_n^{-1}(B_n) \hookrightarrow \mathcal{Z}_n \hookrightarrow \mathcal{L}_n \\
& & \parallel & & \downarrow q & & \downarrow f_n \\
0 & \longrightarrow & \mathcal{Z}_{n+1} & \longrightarrow & \mathcal{G}_{n+1} & \xrightarrow{\partial_{n+1}} & B_n \hookrightarrow \mathcal{Z}_n \hookrightarrow \mathcal{G}_n
\end{array}$$

In this diagram (1) Γ_{n+1} is the pullback of (∂_{n+1}, f'_n) and (2) we chose a surjective map $\varphi : \mathcal{L}_{n+1} \twoheadrightarrow \Gamma_{n+1}$ with $\mathcal{L}_{n+1} \in \mathcal{V}(X)$. We have the following: (1) By the properties of pull back p, q are surjective and $\ker(p) \cong \mathcal{Z}_{n+1}$. (2) Let $g_n = q\varphi$ and the differential d_{n+1} is defined as shown. Denote $\mathcal{Z}_{n+1} := \ker(d_{n+1})$. (3) Since φ, p are surjective, $\text{Image}(d_{n+1}) = f_n^{-1}(B_n)$. (4) Since f_n is surjective, the homology map $\mathcal{H}(g_n) : \mathcal{H}_n(\mathcal{L}_\bullet) \rightarrow \mathcal{H}_n(\mathcal{G}_\bullet)$ is surjective. It also follows that $\mathcal{H}(g_n)$ is injective. Therefore, $\mathcal{H}(g_n) : \mathcal{H}_n(\mathcal{L}_\bullet) \xrightarrow{\sim} \mathcal{H}_n(\mathcal{G}_\bullet)$ is an isomorphism. (5) By Snake lemma, it also follows that f_{n+1} is surjective. This completes the proof of the first part of the lemma.

Finally, if all the homologies $\mathcal{H}_i(\mathcal{G}_\bullet) \in \mathcal{B}(X)$, since $\mathcal{B}(X)$ is an exact category and every epimorphism in $\mathcal{B}(X)$ is admissible, $\mathcal{Z}_i(\mathcal{L}_\bullet), B_i(\mathcal{L}_\bullet) \in \mathcal{B}(X)$. The left tail of \mathcal{L}_\bullet becomes a resolution of $B_n(\mathcal{L}_\bullet)$ for some $n \gg 0$. Since $\mathcal{G}_i = 0 \ \forall i \gg 0$, we can truncate \mathcal{L}_\bullet . The proof is complete. \square

Now, we proceed to establish that given a morphism $g : \mathcal{F} \rightarrow \mathcal{G}$ of coherent sheaves, and resolutions $\mathcal{L}_\bullet, \mathcal{E}_\bullet$, respectively, of \mathcal{F}, \mathcal{G} , g lifts to a morphism $\mathcal{L}_\bullet \rightarrow \mathcal{E}_\bullet$ in the derived category.

Lemma 3.3. *Suppose X is a noetherian scheme, as in (2.1). Let $g : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of coherent sheaves and $\mathcal{F}_\bullet, \mathcal{G}_\bullet$ be complexes in $\text{Ch}^{\geq 0}(\text{Coh}(X))$ such that*

$$\mathcal{H}_0(\mathcal{F}_\bullet) = \mathcal{F}, \quad \mathcal{H}_0(\mathcal{G}_\bullet) = \mathcal{G}, \quad \text{and} \quad \mathcal{H}_i(\mathcal{F}_\bullet) = \mathcal{H}_i(\mathcal{G}_\bullet) = 0 \ \forall i \neq 0,$$

Consider \mathcal{G} as a complex in $\text{Ch}^{\geq 0}(\text{Coh}(X))$ concentrated at degree zero and let Γ_\bullet be the pullback:

$$\begin{array}{ccc}
\Gamma_\bullet & \xrightarrow{t} & \mathcal{F}_\bullet \\
\downarrow G & & \downarrow \\
\mathcal{G}_\bullet & \longrightarrow & \mathcal{G}
\end{array}
\quad \text{Then, } \exists \text{ a commutative diagram} \quad
\begin{array}{ccccc}
\mathcal{F}_\bullet & \xleftarrow{t} & \Gamma_\bullet & \xrightarrow{G} & \mathcal{G}_\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F} & \xlongequal{\quad} & \mathcal{F} & \xrightarrow{g} & \mathcal{G}
\end{array}$$

such that G, t are morphisms in $\text{Ch}^{\geq 0}(\text{Coh}(X))$ and $t : \Gamma_\bullet \rightarrow \mathcal{F}_\bullet$ is a quasi-isomorphism.

Proof. We establish the notations as in the diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{F}_r & \xrightarrow{d_r} & \mathcal{F}_{r-1} & \xrightarrow{d_{r-1}} & \cdots \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F} \longrightarrow 0 \\
& & & & & & & \downarrow g \\
\cdots & \longrightarrow & \mathcal{G}_r & \xrightarrow{\partial_r} & \mathcal{G}_{r-1} & \longrightarrow & \cdots \longrightarrow \mathcal{G}_0 \xrightarrow{\partial_0} \mathcal{G} \longrightarrow 0
\end{array}$$

where the horizontal lines are exact. Recall that the pullback of complexes is constructed by taking degree wise pullback. So, $\Gamma_i = 0 \ \forall \ i < 0$. At degree $i = 0$, of the pullback diagram is given by:

$$\begin{array}{ccc}
\Gamma_0 & \xrightarrow{t_0} & \mathcal{F}_0 \\
G_0 \downarrow & & \downarrow gd_0 \\
\mathcal{G}_0 & \xrightarrow{\partial_0} & \mathcal{G}
\end{array}
\quad \text{Since } \partial_0 \text{ is surjective, so is } t_0.$$

Therefore, $\partial'_0 := gt_0 : \Gamma_0 \twoheadrightarrow \mathcal{F}$ is surjective. We have $\ker(\partial'_0) = \ker(d_0) \oplus \ker(\partial_0)$. However, $\forall \ i > 0 \ \Gamma_i = \mathcal{F}_i \oplus \mathcal{G}_i$ and the differentials $\partial'_i = d_i \oplus \partial_i : \Gamma_i \longrightarrow \Gamma_{i-1}$. The proof is complete. \square

Corollary 3.4. *Suppose X is a noetherian scheme, as in (2.1). Let \mathcal{F} be an object in $\text{Coh}(X)$ and $\mathcal{F}_\bullet, \mathcal{F}'_\bullet$ be two resolutions of \mathcal{F} in the chain complex category $\text{Ch}^{\geq 0}(X)$. Then $\mathcal{F}_\bullet \cong \mathcal{F}'_\bullet$ in the derived category $D^+(\text{Coh}(X))$.*

Proof. It follows immediately, by an application of corollary 3.3, with $g = 1_{\mathcal{F}}$, since in this case G is also a quasi-isomorphism. \square

Corollary 3.5. *Suppose X is a noetherian scheme, as in (2.1). Under the hypotheses and notations in lemma 3.3, there is a complex $\mathcal{L}_\bullet \in \text{Ch}^{\geq 0}(\mathcal{V}(X))$, a quasi-isomorphism $\tau : \mathcal{L}_\bullet \longrightarrow \mathcal{F}_\bullet$, and a morphism $\gamma : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ such that the diagram*

$$\begin{array}{ccccc}
\mathcal{F}_\bullet & \xleftarrow{\tau} & \mathcal{L}_\bullet & \xrightarrow{\gamma} & \mathcal{G}_\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F} & \xlongequal{\quad} & \mathcal{F} & \xrightarrow{g} & \mathcal{G}
\end{array}
\quad \text{commutes.}$$

Further, if $\mathcal{F} \in \mathcal{B}(X)$ and $\mathcal{G}_\bullet \in \text{Ch}^b(\text{Coh}(X))$, then \mathcal{L}_\bullet can be chosen to be in $\text{Ch}^b(\mathcal{V}(X))$.

Proof. We use all the notations as in lemma 3.3. By lemma 3.1, applied to the identity morphism $Id : \mathcal{F} \longrightarrow \mathcal{F}$ and Γ_\bullet , there a complex $\mathcal{L}_\bullet \in \text{Ch}^+(\mathcal{V}(X))$ and a quasi-isomorphism $\epsilon : \mathcal{L} \longrightarrow \Gamma_\bullet$. Now, the corollary is established, with the choices $\tau = t\epsilon$ and $\gamma = G\epsilon$. The last statement also follows from the corresponding statement in lemma 3.1. The proof is complete. \square

For morphisms $g : \mathcal{F} \longrightarrow \mathcal{G}$ in $\text{Coh}(X)$, a "lift" in the derived category is defined as follows.

Definition 3.6. Suppose X is a noetherian scheme, in (2.1). Let $g : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of coherent sheaves on X . Suppose $\mathcal{E}_\bullet \rightarrow \mathcal{F}, \mathcal{Q}_\bullet \rightarrow \mathcal{G}$ are two resolutions, where $\mathcal{E}_\bullet, \mathcal{Q}_\bullet$ are in $Ch^{\geq 0}(\mathcal{V}(X))$. By corollary 3.5, there is a complex $\mathcal{L}_\bullet \in Ch^{\geq 0}(\mathcal{V}(X))$, a quasi-isomorphism $\tau : \mathcal{L}_\bullet \rightarrow \mathcal{E}_\bullet$, and a morphism $\gamma : \mathcal{L}_\bullet \rightarrow \mathcal{Q}_\bullet$ such that the diagram

$$\begin{array}{ccccc} \mathcal{E}_\bullet & \xleftarrow{\tau} & \mathcal{L}_\bullet & \xrightarrow{\gamma} & \mathcal{Q}_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F} & \xlongequal{\quad} & \mathcal{F} & \xrightarrow{g} & \mathcal{G} \end{array} \quad \text{commutes. Define } \zeta(g) : \mathcal{E}_\bullet \rightarrow \mathcal{Q}_\bullet \text{ by } \zeta(g) = \gamma\tau^{-1}$$

as a morphism $Mor_{D^+(\mathcal{V}(X))}(\mathcal{E}_\bullet, \mathcal{Q}_\bullet)$ in the derived category. Further, if $\mathcal{E}_\bullet, \mathcal{Q}_\bullet \in Ch^b(\mathcal{V}(X))$ then $\zeta(g)$ is defined in $D^b(\mathcal{V}(X))$.

We now establish that $\zeta(g)$ is well defined.

Lemma 3.7. Under the set up of definition 3.6, $\zeta(g)$ is well defined in $D^+(\mathcal{V}(X))$. Further, $\zeta(g)$ is in $D^b(\mathcal{V}(X))$, if $\mathcal{E}_\bullet, \mathcal{Q}_\bullet \in Ch^b(\mathcal{V}(X))$.

Proof. As in (3.3), consider \mathcal{G} as a complex in $Ch^+(Coh(X))$ and let Γ_\bullet be the pullback:

$$\begin{array}{ccc} \Gamma_\bullet & \xrightarrow{t} & \mathcal{E}_\bullet \\ \downarrow G & & \downarrow \\ \mathcal{Q}_\bullet & \longrightarrow & \mathcal{G} \end{array} \quad \text{Then, the diagram} \quad \begin{array}{ccccc} \mathcal{E}_\bullet & \xleftarrow{t} & \Gamma_\bullet & \xrightarrow{G} & \mathcal{Q}_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F} & \xlongequal{\quad} & \mathcal{F} & \xrightarrow{g} & \mathcal{G} \end{array} \quad \text{commutes}$$

and t is a quasi-isomorphism. Given two pairs $(\mathcal{L}_\bullet, \tau, \gamma)$ and $(\mathcal{L}'_\bullet, \tau', \gamma')$ as in definition 3.6, from the properties of the pullback it follows, that there are maps $\epsilon : \mathcal{L}_\bullet \rightarrow \Gamma_\bullet$, $\epsilon' : \mathcal{L}'_\bullet \rightarrow \Gamma_\bullet$ such that the diagrams

$$\begin{array}{ccccc} & & \mathcal{E}_\bullet & & \\ & \nearrow \tau & \uparrow t & \searrow & \\ \mathcal{L}_\bullet & \xrightarrow{\epsilon} & \Gamma_\bullet & \xrightarrow{G} & \mathcal{G} \\ & \searrow \gamma & \downarrow & \nearrow & \\ & & \mathcal{Q}_\bullet & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \mathcal{E}_\bullet & & \\ & \nearrow \tau' & \uparrow t & \searrow & \\ \mathcal{L}'_\bullet & \xrightarrow{\epsilon'} & \Gamma_\bullet & \xrightarrow{G} & \mathcal{G} \\ & \searrow \gamma' & \downarrow & \nearrow & \\ & & \mathcal{Q}_\bullet & & \end{array} \quad \text{commute.}$$

Note that ϵ, ϵ' are quasi-isomorphisms. By Ore condition (see proof of [W, 10.4.1]) in $Coh^+(X)$ there is $\Delta_\bullet \in Ch^+(Coh(X))$ and quasi-isomorphisms $\mu : \Delta_\bullet \rightarrow \mathcal{L}_\bullet$ and $\mu' : \Delta_\bullet \rightarrow \mathcal{L}'_\bullet$ such that $\epsilon\mu = \epsilon'\mu'$. By (3.2), there is quasi-isomorphism $\nu : \mathcal{U}_\bullet \rightarrow \Delta_\bullet$, with $\mathcal{U}_\bullet \in Ch^+(\mathcal{V}(X))$.

With $\eta = \mu\nu$, $\eta' = \mu'\nu$, the diagram

$$\begin{array}{ccccc} & & \mathcal{L}_\bullet & & \\ & \nearrow \eta & \uparrow \mu & \searrow \epsilon & \\ \mathcal{U}_\bullet & \xrightarrow{\nu} & \Delta_\bullet & \xrightarrow{\quad} & \Gamma_\bullet \\ & \searrow \eta' & \downarrow \mu' & \nearrow \epsilon' & \\ & & \mathcal{L}'_\bullet & & \end{array} \quad \text{commutes.}$$

All the morphisms in this latter diagram are quasi-isomorphisms. It follows,

$$\tau\eta = \tau'\eta', \quad \gamma\eta = \gamma'\eta'. \quad \text{Therefore, the diagram}$$

$$\begin{array}{ccccc} & & \mathcal{L}_\bullet & & \\ & \swarrow \tau & \uparrow \eta & \searrow \gamma & \\ \mathcal{E}_\bullet & \xleftarrow{\tau\eta} & \mathcal{U}_\bullet & \xrightarrow{\gamma\eta} & \mathcal{Q}_\bullet \\ & \swarrow \tau' & \downarrow \eta' & \searrow \gamma' & \\ & & \mathcal{L}'_\bullet & & \end{array}$$

commutes. So, $\gamma\tau^{-1} = \gamma'(\tau')^{-1}$. This establishes that $\zeta(g)$ is well defined in $D^+(\mathcal{V}(X))$. Similarly, it follows that $\zeta(g)$ is well defined in $D^b(\mathcal{V}(X))$, if $\mathcal{E}_\bullet, \mathcal{Q}_\bullet$ are in $Ch^b(\mathcal{V}(X))$. The proof is complete. \square

In deed, as in the affine case, $\zeta(g)$ is the unique lift of g , in the following sense.

Theorem 3.8. *Consider the set up as in definition 3.6. Then, $\zeta(g) : \mathcal{E}_\bullet \rightarrow \mathcal{Q}_\bullet$ is the unique morphism $\eta : \mathcal{E}_\bullet \rightarrow \mathcal{Q}_\bullet$ such that $\mathcal{H}_0(\eta) = g$.*

Proof. Suppose η is as in the statement of the corollary. Then, $\eta = \gamma'(\tau')^{-1}$, where τ', γ' is given by the commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_\bullet & \xleftarrow{\tau'} \mathcal{L}'_\bullet & \xrightarrow{\gamma'} \mathcal{Q}_\bullet \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{g} & \mathcal{G} \end{array} \quad \text{where } \mathcal{L}'_\bullet \in Ch^+(\mathcal{V}(X)), \text{ and } \tau' \text{ is a quasi-isomorphism.}$$

By replacing sheaves at the negative degrees of \mathcal{L}'_\bullet by zero and \mathcal{L}'_0 by $\ker(\mathcal{L}'_0 \rightarrow \mathcal{L}'_{-1})$, we obtain a complex \mathcal{L}_\bullet and a quasi-isomorphism $\epsilon : \mathcal{L}_\bullet \rightarrow \mathcal{L}'_\bullet$, such that (1) \mathcal{L}_\bullet fits in the diagram of the definition 3.6, with $\tau = \tau'\epsilon$ and $\gamma = \gamma'\epsilon$. It follows, $\zeta(g) = \gamma\tau^{-1} = \gamma'(\tau')^{-1}$. The proof is complete. \square

Now, ζ behaves in a functorial manner, for the morphisms, as follows.

Proposition 3.9. *Let X be a noetherian scheme as in (2.1). Let $g_0 : \mathcal{F}_0 \rightarrow \mathcal{F}_1$ and $g_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be morphisms of coherent sheaves on X . For $i = 0, 1, 2$ let $\mathcal{E}_\bullet^i \twoheadrightarrow \mathcal{F}_i$ be resolutions of \mathcal{F}_i , with \mathcal{E}_\bullet^i in $Ch^{\geq 0}(\mathcal{V}(X))$ (resp. in $Ch^b(\mathcal{V}(X))$). Then $\zeta(g_1 g_0) = \zeta(g_1) \zeta(g_0)$, where $\zeta(g_0), \zeta(g_1), \zeta(g_1 g_0)$ are as defined in (3.6).*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} & & \mathcal{E}_\bullet & & & & \\ & \swarrow \tau & & \searrow \gamma & & & \\ \mathcal{E}_\bullet^0 & \xleftarrow{\tau_0} \mathcal{L}_\bullet^0 & \xrightarrow{\gamma_0} \mathcal{E}_\bullet^1 & \xleftarrow{\tau_1} \mathcal{L}_\bullet^1 & \xrightarrow{\gamma_1} \mathcal{E}_\bullet^2 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathcal{F}_0 & \xlongequal{\quad} \mathcal{F}_0 & \xrightarrow{g_0} \mathcal{F}_1 & \xlongequal{\quad} \mathcal{F}_1 & \xrightarrow{g_1} \mathcal{F}_2 & & \end{array}$$

In this diagram, $(\mathcal{L}_\bullet^i, \tau_i, \gamma_i)$ are obtained by corollary 3.5, for $i = 0, 1$, corresponding to g_i , where \mathcal{L}_\bullet^i are in $Ch^{\geq 0}(\mathcal{V}(X))$ and τ_i are quasi-isomorphism. Further, $(\mathcal{E}, \tau, \gamma)$ is obtained by application of the Ore condition in $Ch^+(\mathcal{V}(X))$ (resp. in $Ch^b(\mathcal{V}(X))$), where τ is a quasi-isomorphism. By (3.8), it follows, $\zeta(g_1 g_0) =$

$$(\gamma_1 \gamma)(\tau_0 \tau)^{-1} = (\gamma_1 \tau_1^{-1})(\tau_1 \gamma)(\tau_0 \tau)^{-1} = \zeta(g_1)(\gamma_0 \tau)(\tau_0 \tau)^{-1} = \zeta(g_1) \gamma_0 \tau_0^{-1} = \zeta(g_1) \zeta(g_0).$$

The proof is complete. \square

We are finally ready to give a complete definition of the functors $\zeta : Coh(X) \longrightarrow D^+(\mathcal{V}(X))$, and $\zeta : \mathcal{B}(X) \longrightarrow D^b(\mathcal{V}(X))$, as follows.

Definition 3.10. *Let X be a noetherian scheme, as in (2.1). Define $\zeta : \mathcal{B}(X) \longrightarrow D^b(\mathcal{V}(X))$ as follows*

1. *For each object $\mathcal{F} \in \mathcal{B}(X)$, by (3.1), choose a resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$ where \mathcal{E}_\bullet is in $Ch^b(\mathcal{V}(X))$. Define*

$$\zeta(\mathcal{F}) := \mathcal{E}_\bullet \text{ in } D^b(\mathcal{V}(X)).$$

For simplicity, we make a convention that if the length of a minimal $\mathcal{V}(X)$ -resolution of \mathcal{F} is r , then by choice $\zeta(\mathcal{F})_i = 0 \forall i > r$.

2. *For morphisms $g : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathcal{B}(X)$, define $\zeta(g) : \zeta(\mathcal{F}) \longrightarrow \zeta(\mathcal{G})$ as in 3.6.*

The functor, $\zeta : Coh(X) \longrightarrow D^+(\mathcal{V}(X))$ is defined similarly.

It is immediate from proposition 3.9 that the functors ζ defined above are well defined.

3.1 The Resolving category and fomalism

Much of the arguments above can be formulated in the realm of resolving subcategories of abelian categories as follows. Before we proceed, we define resolving subcategories of abelian categories.

Definition 3.11. Suppose \mathcal{C} is an abelian category. An exact subcategory \mathcal{V} of \mathcal{C} is called a **resolving subcategory** if, (1) \mathcal{V} is closed under direct summand and direct sum, (2) every epimorphism \mathcal{V} is admissible, and (3) given any object M in \mathcal{C} there is an epimorphism $\mathcal{E} \twoheadrightarrow M$, for some $\mathcal{E} \in \mathcal{V}$.

Remark 3.12. Suppose \mathcal{V} is a resolving subcategory of an abelian category \mathcal{C} . Following are some immediate and obvious comments:

(1) Any object M in \mathcal{C} has a resolution by objects in \mathcal{V} . (2) The category of chain complexes $Ch^b(\mathcal{V})$, $Ch^+(\mathcal{V})$, $Ch(\mathcal{V})$ of objects in \mathcal{V} and the homotopy categories $K^b(\mathcal{V})$, $K^+(\mathcal{V})$, and $K(\mathcal{V})$ are defined, as usual. (3) The derived categories $D^b(\mathcal{V})$, $D^+(\mathcal{V})$, and $D(\mathcal{V})$ are defined by inverting the quasi-isomorphisms in the corresponding homotopy category (*without any regard to any other structure*). Consult the proof of [W, 10.4.1] on localization, which makes sense in our context.

Proposition 3.13. Suppose \mathcal{C} is an abelian category and \mathcal{V} is a resolving subcategory.

Let $\mathcal{B} := \mathcal{B}(\mathcal{V}) := \{\mathcal{F} \in \mathcal{C} : \mathcal{F} \text{ has a finite } \mathcal{V}\text{-resolution}\}$ be the full subcategory.

Assume, given an object $\mathcal{F} \in \mathcal{B}$ and a \mathcal{V} -resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$, the cycle objects $Z_n := \ker(\mathcal{E}_n \rightarrow \mathcal{E}_{n-1}) \in \mathcal{V} \quad \forall n \gg 0$.

1. \mathcal{B} is an exact subcategory and every epimorphism in \mathcal{B} is admissible.
2. There is a natural functor

$$\zeta : \mathcal{B}(\mathcal{V}) \longrightarrow D^b(\mathcal{V}) \quad \text{as defined in (3.6).}$$

Likewise, there is a functor $\zeta : \mathcal{C} \longrightarrow D^+(\mathcal{P})$ (without any condition on Z_n).

3. All of § 3 remain valid, if we replace $Coh(X)$ by \mathcal{C} , $\mathcal{V}(X)$ by \mathcal{V} and $\mathcal{B}(X)$ by $\mathcal{B}(\mathcal{V})$.

Proof. The statements (5b, 3) follows as in the cases of noetherian schemes. We need to give a proof of (1). Let $0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} C \longrightarrow 0$ be an exact sequence in \mathcal{C} . Inductively, a resolution (possibly infinite) of this sequence in $Ch^{\geq 0}(\mathcal{V})$, as follows. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\ & & \downarrow \partial_0 & & \downarrow \varphi_0 & & \parallel & & \\ 0 & \longrightarrow & K & \longrightarrow & \Gamma_0 & \xrightarrow{\gamma_0} & Q_0 & \longrightarrow & 0 \\ & & \parallel & & \downarrow d'_0 & & \downarrow d_0 & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & M & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

In this diagram, $d_0 : Q_0 \twoheadrightarrow C$ is a surjective morphism, is the pullback of (d_0, g) and $\varphi : P_0 \twoheadrightarrow \Gamma_0$ is a surjective morphism, with $Q_0, P_0 \in \mathcal{V}$. Also, $L_0 := \ker(\gamma_0 \varphi_0) \in \mathcal{V}$. By Snake lemma ∂_0 is surjective. Write $\delta_0 = d'_0 \varphi_0$. Again, by Snake lemma

$0 \longrightarrow \ker(\partial_0) \longrightarrow \ker(\delta_0) \longrightarrow \ker(d_0) \longrightarrow 0$ is exact and the process continues.

Now, if $K, C \in \mathcal{B}(\mathcal{V})$ then, by hypothesis, the resolutions $L_\bullet \twoheadrightarrow K, Q_\bullet \twoheadrightarrow C$ terminate, hence process stops and $P_\bullet \in Ch^b(\mathcal{V})$. So, $\mathcal{B}(\mathcal{V})$ is an exact subcategory. Similarly, every epimorphisms in $\mathcal{B}(\mathcal{V})$ is admissible. The proof is complete. \square

4 Duality Properties

In this section X will denote a noetherian scheme, as in (2.1), with $\dim X = d$. Also, $\mathcal{A}(X)$ will denote the abelian subcategory of $\text{Coh}(X)$, as defined in (2.1). With a view on section 7 on formalism, we take the formal approach to the proofs, as opposed to local. First, we define a duality on $\mathcal{A}(X)$.

Definition 4.1. *Suppose X is a noetherian scheme with $\dim X = d$, as in (2.1). We fix a choice of an injective resolution \mathcal{I}_\bullet of \mathcal{O}_X . For a coherent sheaf \mathcal{F} , and integers $n \geq 0$, we define $\mathcal{E}xt^n(\mathcal{F}, \mathcal{O}_X) := \mathcal{H}^n(\mathcal{F}, \mathcal{I}_\bullet)$.*

Lemma 4.2. *Suppose X is a noetherian scheme with $\dim X = d$ and $\mathcal{A}(X)$ is as in (2.1). Define a functor ${}^\vee : \mathcal{A} \rightarrow \mathcal{A}$ by $\mathcal{F}^\vee := \mathcal{E}xt^d(\mathcal{F}, \mathcal{O}_X)$. Then, ${}^\vee$ defines a duality on \mathcal{A} .*

Proof. We need to establish that, for objects $\mathcal{F} \in \mathcal{A}(X)$, there is a natural isomorphism $\tilde{\omega} : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$. Let $\mathcal{E}_\bullet := \zeta(\mathcal{F})$ as defined in (3.10). We denote this complex as

$$0 \longrightarrow \mathcal{E}_d \xrightarrow{\partial_d} \cdots \longrightarrow \mathcal{E}_1 \xrightarrow{\partial_1} \mathcal{E}_0 \xrightarrow{\partial_0} \mathcal{F} \longrightarrow 0$$

So, there is a natural isomorphism $\text{co ker}(\partial_d^*) \xrightarrow{\sim} \mathcal{F}^\vee$ (see [H]). Since $\mathcal{F} \in \mathcal{A}(X)$ we have $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i \neq d$. Therefore, the dual $\mathcal{E}^\#$ yields a resolution:

$$0 \longrightarrow \mathcal{E}_0^* \longrightarrow \cdots \longrightarrow \mathcal{E}_{d-1}^* \longrightarrow \mathcal{E}_d^* \longrightarrow \mathcal{F}^\vee \longrightarrow 0$$

Dualizing again, we have the following diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_d & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow \text{ev} & & & & \downarrow \text{ev} & & \downarrow \text{ev} & & \downarrow \wr \varpi_0 & & \\ 0 & \longrightarrow & \mathcal{E}_d^{**} & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}_1^{**} & \longrightarrow & \mathcal{E}_0^{**} & \longrightarrow & \text{co ker}(\partial_0^{**}) & \longrightarrow & 0 \end{array}$$

The isomorphism ϖ_0 is induced by the evaluation maps. There is also a natural isomorphism $\omega_1 : \text{co ker}(\partial_0^{**}) \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$. Hence, $\tilde{\omega} := \varpi_1 \varpi_0 : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$ is a natural isomorphism. The proof is complete. \square

The derived category $D^b(\mathcal{V}(X))$ has a triangulated structure, with the duality induced by the duality $-^* := \mathcal{H}om(-, \mathcal{O}_X)$, which we will denote by $\#$. We have particular interest in the subcategory $D_{\mathcal{A}}^b(\mathcal{V}(X))$. It was pointed out in ([MS1]) that $D_{\mathcal{A}}^b(\mathcal{V}(X))$ may not be closed under cone construction, but it was stable under duality. This works for any noetherian scheme, as follows.

Lemma 4.3. *Suppose X is a noetherian scheme, with $\dim X = d$ (the condition in (2.1) that every coherent sheaf on X is quotient of a sheaf in $\mathcal{V}(X)$, is not needed.) Also,*

$\mathcal{A} := \mathcal{A}(X)$ be as in (2.1). Let \mathcal{E}_\bullet be a complex in $Ch_{\mathcal{A}}^b(\mathcal{V}(X))$. Then, for all integers $r \in \mathbb{Z}$, there are natural isomorphism

$$\mathcal{E}xt^i\left(\frac{\mathcal{E}_r}{B_r}, \mathcal{O}_X\right) \xrightarrow{\sim} \begin{cases} \mathcal{E}xt^d(\mathcal{H}_{r+i-d}(\mathcal{E}_\bullet), \mathcal{O}_X) & 1 \leq i \leq d \\ 0 & i \geq d+1 \end{cases} \quad (1)$$

where $B_r \subseteq \mathcal{E}_r$ denote the boundary, as in (2.1).

Proof. We assume $\mathcal{E}_r = 0 \ \forall r < 0$. If $\mathcal{H}_0(\mathcal{E}_\bullet) = 0$ then there is nothing to prove. So, assume the $\mathcal{H}_0(\mathcal{E}_\bullet) \neq 0$. Then, $\frac{\mathcal{E}_0}{B_0} = \mathcal{H}_0(\mathcal{E}_\bullet)$ and equation (1) holds. Now assume that equation (1) holds for degree r . We prove it for degree $r+1$. We have two exact sequences

$$0 \longrightarrow B_r \longrightarrow \mathcal{E}_r \longrightarrow \frac{\mathcal{E}_r}{B_r} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{H}_{r+1} \longrightarrow \frac{\mathcal{E}_{r+1}}{B_{r+1}} \longrightarrow B_r \longrightarrow 0$$

The long exact sequence of the first exact sequence yields the following isomorphisms:

$$\forall i \geq 1 \quad \mathcal{E}xt^i(B_r, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{E}xt^{i+1}\left(\frac{\mathcal{E}_r}{B_r}, \mathcal{O}_X\right) \xrightarrow{\sim} \begin{cases} \mathcal{E}xt^d(\mathcal{H}_{r+(i+1)-d}, \mathcal{O}_X) & 1 \leq i \leq d-1 \\ 0 & i \geq d \end{cases}$$

Whether $\mathcal{H}_{r+1} = 0$ or $\mathcal{H}_{r+1} \neq 0$, $\mathcal{E}xt^i(\mathcal{H}_{r+1}, \mathcal{O}_X) = 0 \ \forall i \neq d$. It follows from the second exact sequence

$$\mathcal{E}xt^i\left(\frac{\mathcal{E}_{r+1}}{B_{r+1}}, \mathcal{O}_X\right) \xrightarrow{\sim} \begin{cases} \mathcal{E}xt^i(B_r, \mathcal{O}_X) & \text{if } i \leq d-1 \\ \mathcal{E}xt^d(\mathcal{H}_{r+1}, \mathcal{O}_X) & \text{if } i = d \\ 0 & \text{if } i \geq d+1 \end{cases}$$

This establishes the lemma. \square

Theorem 4.4. Let X be a noetherian scheme as in (4.3) and \mathcal{E}_\bullet be a complex in $Ch_{\mathcal{A}}^b(\mathcal{V}(X))$. Then, for $r \in \mathbb{Z}$, there is a canonical isomorphism $\eta_{\mathcal{E}_\bullet} : \mathcal{H}_{-r}(\mathcal{E}_\bullet^\#) \xrightarrow{\sim} \mathcal{H}_{-d}(\mathcal{E}_\bullet)^\vee$. In particular, $Ch_{\mathcal{A}}^b(\mathcal{V}(X))$ is stable under duality. Further, $\eta_{\mathcal{E}_\bullet}$ is natural with respect to morphisms $f : \mathcal{E}_\bullet \rightarrow \mathcal{E}'_\bullet$ in $Ch_{\mathcal{A}}^b(\mathcal{V}(X))$.

Proof. First, we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \left(\frac{\mathcal{E}_{r-1}}{B_{r-1}}\right)^* & \longrightarrow & (\mathcal{E}_{r-1})^* & \longrightarrow & (B_{r-1})^* \longrightarrow \mathcal{E}xt^1\left(\frac{\mathcal{E}_{r-1}}{B_{r-1}}, \mathcal{O}_X\right) \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \wr \\ 0 & \longrightarrow & \left(\frac{\mathcal{E}_{r-1}}{B_{r-1}}\right)^* & \longrightarrow & (\mathcal{E}_{r-1})^* & \longrightarrow & \left(\frac{\mathcal{E}_r}{B_r}\right)^* \longrightarrow H_{-r}(\mathcal{E}_\bullet^*) \longrightarrow 0 \end{array}$$

The isomorphism ψ is induced. Now, the first part of the theorem follows by composing ψ with the isomorphism, as follows: $\mathcal{E}xt^1\left(\frac{\mathcal{E}_{r-1}}{B_{r-1}}, \mathcal{O}_X\right) \xrightarrow{\sim} \mathcal{E}xt^d(\mathcal{H}_{r-d}, \mathcal{O}_X)$

$$\begin{array}{ccc} \mathcal{E}xt^1\left(\frac{\mathcal{E}_{r-1}}{B_{r-1}}, \mathcal{O}_X\right) & \xrightarrow{\sim} & \mathcal{E}xt^d(\mathcal{H}_{r-d}, \mathcal{O}_X) \\ \downarrow \psi & \nearrow \eta_{\mathcal{E}_\bullet} & \\ H_{-r}(\mathcal{E}_\bullet^\#) & & \end{array}$$

The latter part follows because all the isomorphisms are natural. The proof is complete. \square

5 The Main Isomorphism Theorem

Having established the that $D_{\mathcal{A}}^b(\mathcal{V}(X))$ is stable under duality, we discuss Witt group of $D_{\mathcal{A}}^b(\mathcal{V}(X))$. We set up some basic framework, analogous to ([MS1]).

1. For a complex \mathcal{E}_{\bullet} in $Ch^b(\mathcal{V}(X))$, $T_u(\mathcal{E}_{\bullet})$ will denote the unsigned translation, and $T_s(\mathcal{E}_{\bullet})$ the standard translation which changes the sign of the differential.
2. Denote the shifted Derived categories: $T^n D_{\mathcal{A}}^b(\mathcal{V}(X))_u^{\pm} := (D_{\mathcal{A}}^b(\mathcal{V}(X)), T_u^n o\#, 1, \pm\varpi)$ and $T^n D_{\mathcal{A}}^b(\mathcal{V}(X))_s^{\pm} := (D_{\mathcal{A}}^b(\mathcal{V}(X)), T_s^n o\#, 1, \pm\varpi)$ where ϖ is the evaluation map.
3. As was pointed out, $D_{\mathcal{A}}^b(\mathcal{V}(X))$ may not have a triangulated structure. Readers are also referred to ([MS1]), for a definition of Witt groups of subcategories of triangulated categories with duality. Note that this definition is very similar to that of Witt groups of exact catgories in [B3], requiring the lagrangians to be admissible monomorphisms. As usual, we denote $W^n(D_{\mathcal{A}}^b(\mathcal{V}(X))_u^{\pm}) := W(T^n D_{\mathcal{A}}^b(\mathcal{V}(X))_u^{\pm})$ and $W^n(D_{\mathcal{A}}^b(\mathcal{V}(X))_s^{\pm}) := W(T^n D_{\mathcal{A}}^b(\mathcal{V}(X))_s^{\pm})$.

Now, we prove that the functor $\zeta : \mathcal{A}(X) \longrightarrow D_{\mathcal{A}}^b(\mathcal{V}(X))$ induces isomorphisms of Witt groups. First, we prove ζ induces a homomorphism of Witt groups.

Theorem 5.1. *Suppose X is a noetherian scheme, as in (2.1), with $d = \dim X$. Let $\mathcal{A} := \mathcal{A}(X)$ be as in (2.1). Then, the functor $\zeta : \mathcal{A} \longrightarrow D_{\mathcal{A}}^b(\mathcal{V}(X))$ induces a well defined homomorphism*

$$W(\zeta) : W(\mathcal{A}^{\vee}, \pm\tilde{\varpi}) \longrightarrow W^d(D_{\mathcal{A}}^b(\mathcal{V}(X))_u^{\pm})$$

Proof. We will only give the proof for the $+$ -duality. We first define the homomorphism $W(\zeta) : W(\mathcal{A}^{\vee}, \tilde{\varpi}) \longrightarrow W^d(D_{\mathcal{A}}^b(\mathcal{V}(X))_u^+)$ and then prove that it is well defined. Suppose (\mathcal{F}, φ_0) is a symmetric form in $(\mathcal{A}^{\vee}, \tilde{\varpi})$. Write $\mathcal{X}_{\bullet} := \zeta(\mathcal{F})$. By choice, $\mathcal{X}_i = 0 \ \forall \ i > d$. Then, $\mathcal{X}_{\bullet}^{\#}$ gives a resolution of \mathcal{F}^{\vee} . It is also easy to see that ζ preserves the duality on \mathcal{A} . By the uniqueness (3.8) of the lifts of morphisms in $\mathcal{A}(X)$, the identity $\varphi_0 = \varphi_0^{\vee} \tilde{\varpi}$ produces a symmetric form $\zeta(\varphi_0) : \mathcal{X}_{\bullet} \xrightarrow{\sim} \mathcal{X}_{\bullet}^{\#}$. Therefore, the association $(\mathcal{F}, \varphi_0) \mapsto (\mathcal{X}_{\bullet}, \zeta(\varphi_0))$ induces a homomorphism $MW(\zeta) : MW(\mathcal{A}^{\vee}, \tilde{\varpi}) \longrightarrow MW(T^d(D_{\mathcal{A}}^b(\mathcal{V}(X))_u^+))$ of monoids of symmetric spaces.

Now we want to prove that $MW(\zeta)$ maps neutral spaces to neutral spaces. Suppose (\mathcal{F}, φ_0) is a neutral space in $(\mathcal{A}^{\vee}, \tilde{\varpi})$. So, there is an exact sequence (i.e. a lagrangian) $0 \longrightarrow \mathcal{G} \xrightarrow{\alpha_0} \mathcal{F} \xrightarrow{\alpha_0^{\vee} \varphi_0} \mathcal{G}^{\vee} \longrightarrow 0$. As above, write $\mathcal{X}_{\bullet} := \zeta(\mathcal{F})$. We would like to show $(\mathcal{X}_{\bullet}, \zeta(\varphi_0))$ is neutral. It will be convenient to work with forms without denominators. By

(3.5 or 3.6), $\zeta(\varphi_0) = \gamma\tau^{-1} : \mathcal{X}_\bullet \xrightarrow{\sim} \mathcal{X}_\bullet^\#$, where τ, γ are quasi-isomorphisms in $Ch^b(\mathcal{V}(X))$, $\mathcal{E}_\bullet \in Ch^+(\mathcal{V}(X))$, such that the diagram

$$\begin{array}{ccc} \mathcal{X}_\bullet & \xleftarrow{\tau} \mathcal{E}_\bullet & \xrightarrow{\gamma} \mathcal{X}_\bullet^\# \\ \downarrow & & \downarrow \\ \mathcal{F} & \xlongequal{\quad} \mathcal{F} & \xrightarrow{\varphi_0} \mathcal{F}^\vee \end{array} \quad \text{commutes.} \quad \text{With } \varphi = \tau^\# \gamma \text{ the diagram} \quad \begin{array}{ccc} \mathcal{E}_\bullet & \xrightarrow{\quad} & \mathcal{F} \\ \varphi \downarrow & & \downarrow \varphi_0 \\ \mathcal{E}_\bullet^\# & \xrightarrow{\quad} & \mathcal{F}^\vee \end{array} \quad (2)$$

commutes and φ is a quasi-isomorphism in $Ch^b(\mathcal{V}(X))$, without denominator. Clearly, $(\mathcal{E}_\bullet, \varphi)$ and $(\mathcal{X}_\bullet, \zeta(\varphi_0))$ are isometric.

Now that $MW(\zeta)(\mathcal{F}, \varphi_0)$ is given by a denominator free quasi-isomorphism $\varphi : \mathcal{E}_\bullet \rightarrow \mathcal{E}_\bullet^\#$, the rest of the argument is borrowed from ([B1], *for extra details see* [MS1]). We outline the proof to point out the subtleties. Let $\mathcal{L}_\bullet := \zeta(\mathcal{G})$. By choice, $\mathcal{L}_0 = 0$ unless $d \geq i \geq 0$. Then, $\mathcal{L}_\bullet^\#$ yields a resolution of \mathcal{G}^\vee . Since $\varphi = \zeta(\varphi_0)$, by (3.9), the above exact sequence lifts to an exact sequence $0 \rightarrow \mathcal{L}_\bullet \xrightarrow{\alpha} \mathcal{E}_\bullet \xrightarrow{\alpha^\# \varphi} \mathcal{L}_\bullet^\# \rightarrow 0$. Since the homotopy category $K^b(\mathcal{V}(X))$ has a triangulated structure, we can embed α in an exact triangle in $T^d K^b(\mathcal{V}(X))$ and obtain the following diagram and a morphism s , as follows:

$$\begin{array}{ccccccc} \mathcal{L}_\bullet & \xrightarrow{\alpha} & \mathcal{E}_\bullet & \xrightarrow{j} & \mathcal{V}_\bullet & \xrightarrow{k} & T(\mathcal{L}_\bullet) \\ \downarrow & & \downarrow \alpha^\# \varphi & & \downarrow s & & \downarrow 0 \\ 0 & \longrightarrow & \mathcal{L}_\bullet^\# & \xlongequal{\quad} & \mathcal{L}_\bullet^\# & \longrightarrow & 0. \end{array} \quad \text{in } T^d K^b(\mathcal{V}(X)).$$

From the long exact sequence of homologies, it follows $\mathcal{H}_0(\mathcal{V}_\bullet) \cong \mathcal{G}^\vee$ and $\mathcal{H}_i(\mathcal{V}_\bullet) = 0 \forall i \neq 0$. So, s is an isomorphism in $D_{\mathcal{A}}^b(\mathcal{V}(X))$. Therefore, we get the exact triangle

$$\mathcal{L}_\bullet \xrightarrow{\alpha} \mathcal{E}_\bullet \xrightarrow{\alpha^\# \varphi} \mathcal{L}_\bullet^\# \xrightarrow{ks^{-1}} T(\mathcal{L}_\bullet) \quad \text{in } D_{\mathcal{A}}^b(\mathcal{V}(X)).$$

Translating this triangle, and with $w = -T^{-1}(ks^{-1})$, we get the exact triangle

$$T^{-1}\mathcal{L}_\bullet^\# \xrightarrow{w} \mathcal{L}_\bullet \xrightarrow{\alpha} \mathcal{E}_\bullet \xrightarrow{\alpha^\# \varphi} \mathcal{L}_\bullet^\# \quad \text{in } T^d D_{\mathcal{A}}^b(\mathcal{V}(X)).$$

It remains to show that $T^{-1}w^\# = w$. Routine analysis shows that this is equivalent to showing $T(k^\#)s = T(s^\#)k$. Indeed, $s : \mathcal{V}_n = \mathcal{E}_n \oplus \mathcal{L}_{n-1} \rightarrow \mathcal{L}_{d-n}^*$ is a morphism of complexes. Therefore all the maps in this equation $T(k^\#)s = T(s^\#)k$ are morphisms of complexes. It would suffice to show that this identity holds in $T^d K^b(\mathcal{V}(X))$, which can be done exactly as in ([B1, MS1]). This establishes that ζ induces a homomorphism of the Witt groups. \square

Following lemma will be helpful for subsequent discussion.

Lemma 5.2. *Suppose X is a noetherian scheme with $\dim X = d$. Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric from in $T^d D_{\mathcal{A}}^b(\mathcal{V}(X))$. Assume, for some $n \geq 0$, $\mathcal{H}_{-n}(\mathcal{E}_\bullet) \neq 0$ and $\mathcal{H}_i(\mathcal{E}_\bullet) = 0 \forall i < -n$. Then, $(\mathcal{E}_\bullet, \varphi)$ is isometric, to a symmetric from $(\mathcal{E}'_\bullet, \varphi')$ such that $\mathcal{E}'_i = 0$, unless $n + d \geq i \geq -n$.*

Proof. (We give a proof that applies to § 7.) We use the notations from (2.1). Note the cycle sheaf $\mathcal{Z}_{-n}(\mathcal{E}_\bullet) \in Ch^b(\mathcal{V}(X))$. Replacing, \mathcal{E}_i by zero, when $i < -n$ and \mathcal{E}_{-n} by $\mathcal{Z}_{-n}(\mathcal{E}_\bullet)$, we obtain a complex \mathcal{E}'_\bullet , with $\mathcal{E}'_i = 0 \ \forall i < -n$ and a quasi-isomorphism to $\eta : \mathcal{E}'_\bullet \longrightarrow \mathcal{E}_\bullet$. Therefore, replacing $(\mathcal{E}_\bullet, \varphi)$ by $(\mathcal{E}'_\bullet, \eta^\# \varphi \eta)$, we assume that $\mathcal{E}_i = 0 \ \forall i < -n$.

By the duality theorem 4.4, $\forall k > n$, we have $\mathcal{H}_k(\mathcal{E}_\bullet) \cong \mathcal{H}_k(\mathcal{E}_\bullet^\#) \cong \mathcal{H}_{-k}(\mathcal{E}_\bullet)^\vee = 0$. Inductively, we show that we can assume $\mathcal{E}_i = 0$ for all $i \geq n + d + 1$. We will assume $\mathcal{E}_i = 0$ for all $i \geq n + d + 2$. Write $K := \ker(\partial_{n+d+1}^*)$. Then, $K \in \mathcal{V}(X)$. Dualizing the exact sequences $0 \longrightarrow K \longrightarrow \mathcal{E}_{n+d}^* \xrightarrow{\partial_{n+d+1}^*} \mathcal{E}_{n+d+1}^* \longrightarrow 0$, we get the exact sequence $0 \longrightarrow \mathcal{E}_{n+d+1} \xrightarrow{\partial_{n+d+1}} \mathcal{E}_{n+d} \longrightarrow K^* \longrightarrow 0$. Since $K^* \in \mathcal{V}(X)$, by replacing \mathcal{E}_i by zero for $i \geq n + d + 1$ and \mathcal{E}_{n+d} by K^* , we get a new complex \mathcal{E}'_\bullet and a quasi-isomorphism $\eta : \mathcal{E}_\bullet \longrightarrow \mathcal{E}'_\bullet$. It follows $(\mathcal{E}_\bullet, \varphi)$ is isometric to $(\mathcal{E}'_\bullet, (\eta^{-1})^\# \varphi \eta^{-1})$. The proof is complete. \square

As in ([B1, B3]), the proof of surjectivity of $W(\zeta)$ is done by reduction of support of the symmetric form by application of the sublagrangian theorem ([B2, 4.20]). In the non-affine case, both the construction and the proof that it is a sublagrangian require further finesse. The following lemma will be useful.

Lemma 5.3. *Suppose X is as in (2.1). Let $\mathcal{L}_\bullet, \mathcal{G}_\bullet \in Ch^b(\mathcal{V}(X))$ be complexes and $\eta_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism, as in the diagram*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{L}_n & \longrightarrow & \mathcal{L}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{L}_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow \eta & & \downarrow \eta & & & & \downarrow \eta_0 & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{G}_n & \longrightarrow & \mathcal{G}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{G}_{-1} & \longrightarrow & \cdots \end{array}$$

such that

1. $\mathcal{H}_r(\mathcal{G}_\bullet) = 0 \ \forall r \geq 0$.
2. $\mathcal{H}_r(\mathcal{L}_\bullet) = 0 \ \forall r \neq 0$ and $\mathcal{L}_r = 0 \ \forall r < 0$.

Then, $\eta = 0$ in $D^b(\mathcal{V}(X))$.

Proof. We use the notations in (2.1). Imitating the arguments in (3.7), let Γ_\bullet be the complex: $\cdots \longrightarrow \mathcal{L}_2 \oplus \mathcal{G}_2 \longrightarrow \mathcal{L}_1 \oplus \mathcal{G}_1 \longrightarrow \mathcal{L}_0 \oplus \mathcal{Z}_0(\mathcal{G}_\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$. Let $t : \Gamma_\bullet \longrightarrow \mathcal{L}_\bullet$ and $g : \Gamma_\bullet \longrightarrow \mathcal{G}_\bullet$ be the projection maps. Then, t is a quasi-isomorphism. We have the following commutative diagram of morphisms:

$$\begin{array}{ccccc} & & \mathcal{L}_\bullet & & \\ & \swarrow 1 & \downarrow (1, \eta) & \searrow \eta & \\ \mathcal{L}_\bullet & \xleftarrow{t} & \Gamma_\bullet & \xrightarrow{g} & \mathcal{G}_\bullet \\ & \swarrow 1 & \downarrow (1, 0) & \searrow 0 & \\ & & \mathcal{L}_\bullet & & \end{array}$$

Since, t is quasi-isomorphism, so is $(1, \eta)$. Rest of the arguments in (3.7) works, which uses Ore condition, together with (3.1). Note $\Gamma_\bullet, \mathcal{L}_\bullet$ are resolutions, so good enough to apply arguments in (3.7). The proof is complete. \square

Now we prove that $W(\zeta)$ is surjective, as follows.

Proposition 5.4. *Let X be a noetherian scheme, with $\dim X = d$, as in (2.1). Then, the homomorphism $W(\zeta) : W(\mathcal{A}^\vee, \pm \tilde{\omega}) \longrightarrow W^d(D_{\mathcal{A}}^b(\mathcal{V}(X))_u^\pm)$ is surjective.*

Proof. We point out the subtleties involved here in the non-affine case, beyond the arguments in ([MS1]). We will only consider the case of +duality and the case of skew duality follows similarly. As usual, the proof is done by reducing the length of the width of the forms. Suppose $x = [(\mathcal{E}_\bullet, \varphi)] \in W^d(D_{\mathcal{A}}^b(\mathcal{V}(X))_u)$, represented by the a symmetric form $(\mathcal{E}_\bullet, \varphi)$. First, $\varphi = fs^{-1}$ for some quasi-isomorphism $s : \mathcal{E}_\bullet \longrightarrow \mathcal{E}'_\bullet$ for some $\mathcal{E}'_\bullet \in Ch^b(\mathcal{V}(X))$. Replacing $(\mathcal{E}_\bullet, \varphi)$ by $(\mathcal{E}'_\bullet, s^\# \varphi s)$, we assume that $\varphi : \mathcal{E}_\bullet \longrightarrow \mathcal{E}_\bullet^\#$ is a quasi-isomorphism (without denominator) in $Ch^b(\mathcal{V}(X))$.

Assume, for some $n > 0$, suppose $\mathcal{H}_{-n}(\mathcal{E}_\bullet) \neq 0$ and $\mathcal{H}_i(\mathcal{E}_\bullet) = 0 \forall i < -n$. We will prove that there is a symmetric form $(\mathcal{R}_\bullet, \psi)$ such that $x = [(\mathcal{E}_\bullet, \varphi)] = [(\mathcal{R}_\bullet, \psi)]$ and $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ unless $n-1 \geq i \geq -(n-1)$. By lemma 5.2, we assume that $\mathcal{E}_i = 0$ unless $n+d \geq i \geq -n$. Further, by duality theorem 4.4, $\mathcal{H}_i(\mathcal{E}_\bullet) \cong \mathcal{H}_i(\mathcal{E}_\bullet^\#) \cong \mathcal{H}_{-i}(\mathcal{E}_\bullet)^\vee = 0$ for all $i > d$. So, the left tail \mathcal{E}_\bullet is a resolution of $\frac{\mathcal{E}_n}{B_n(\mathcal{E}_\bullet)}$. By lemma 3.1, the morphism $\mathcal{H}_n(\mathcal{E}_\bullet) \hookrightarrow \frac{\mathcal{E}_n}{B_n(\mathcal{E}_\bullet)}$ induces morphism of complexes, as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}_{n+d} & \longrightarrow & \mathcal{L}_{n+d-1} & \longrightarrow & \cdots \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{H}_n(\mathcal{E}_\bullet) \longrightarrow 0 \\ & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu_n & \downarrow \nu_n \\ 0 & \longrightarrow & \mathcal{E}_{n+d} & \longrightarrow & \mathcal{E}_{n+d-1} & \longrightarrow & \cdots \longrightarrow \mathcal{E}_n \longrightarrow \frac{\mathcal{E}_n}{B_n(\mathcal{E}_\bullet)} \longrightarrow 0 \end{array}$$

where $\mathcal{L}_i \in \mathcal{V}(X)$ and both the lines are exact. Since ν_n maps to $\mathcal{Z}_n(\mathcal{E}_\bullet)$, with $\mathcal{L}_i = 0 \forall i \leq n-1$, it extends to morphism $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ in $Ch^b(\mathcal{V}(X))$. We claim ν is a sublagrangian of $(\mathcal{E}_\bullet, \varphi)$. To see this, write $\eta = \nu^\# \varphi \nu : \mathcal{L}_\bullet \longrightarrow \mathcal{L}_\bullet^\#$. We can write η as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}_{n+d} & \longrightarrow & \mathcal{L}_{n+d-1} & \longrightarrow & \cdots \longrightarrow \mathcal{L}_n \longrightarrow 0 \longrightarrow \cdots \\ & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta_n & \downarrow \eta_n \\ 0 & \longrightarrow & \mathcal{L}_{-n}^\# & \longrightarrow & \mathcal{L}_{-(n-1)}^\# & \longrightarrow & \cdots \longrightarrow \mathcal{L}_{d-n}^\# \longrightarrow \mathcal{L}_{d-n+1}^\# \longrightarrow \cdots \end{array}$$

Since, only at degree $-n$, $\mathcal{L}_\bullet^\#$ has a non zero homology, the second line is exact at degrees $i \geq n$. By lemma 5.3, $\eta = 0$ in $D^b(\mathcal{V}(X))$. Rest of the arguments in [MS1] works, which we outline briefly for completeness. As in [B2, 4.20], embed ν in an exact triangle as in

the first line in the diagram (3) and complete the commutative diagram

$$\begin{array}{ccccccc}
 T^{-1}\mathcal{N}_\bullet & \xrightarrow{\nu_0} & \mathcal{L}_\bullet & \xrightarrow{\nu} & \mathcal{E}_\bullet & \xrightarrow{\nu_1} & \mathcal{N}_\bullet \\
 \downarrow T^{-1}\mu_0^\# & & \downarrow \mu_0 & & \downarrow \varphi & & \downarrow \mu_0^\# \\
 T^{-1}\mathcal{L}_\bullet^\# & \longrightarrow & \mathcal{N}_\bullet^\# & \longrightarrow & \mathcal{E}_\bullet^\# & \longrightarrow & \mathcal{L}_\bullet^\# \\
 & & \downarrow \mu_1 & & & & \\
 & & \mathcal{R}_\bullet & & & & \\
 & & \downarrow \mu_2 & & & & \\
 & & T\mathcal{L}_\bullet & & & &
 \end{array} \tag{3}$$

where the second line the dual of the first line, by choice μ_0 is "very good" (see [B2] for definition and existence) and \mathcal{R}_\bullet is the cone of μ_0 . We consider this as a diagram in $D^b(\mathcal{V}(X))$, which is a triangulated category. By [B2, 4.20] there is a symmetric form $\psi : \mathcal{R}_\bullet \xrightarrow{\sim} \mathcal{R}_\bullet^\#$ such that $(\mathcal{R}_\bullet, -\psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$ in $D^b(\mathcal{V}(X))$. This is shown by exhibiting a lagrangian $\lambda : \mathcal{N}_\bullet^\# \longrightarrow (\mathcal{E}_\bullet, -\varphi) \perp (\mathcal{R}_\bullet, \psi)$. We would have to show that \mathcal{R}_\bullet is in $D_{\mathcal{A}}^b(\mathcal{V}(X))$ and λ is a lagrangian in $D_{\mathcal{A}}^b(\mathcal{V}(X))$. It suffices to show,

1. $\mathcal{N}_\bullet, \mathcal{N}_\bullet^\#$ and \mathcal{R}_\bullet are in $D_{\mathcal{A}}^b(\mathcal{V}(X))$,
2. and $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ unless $n - 1 \geq i \geq -(n - 1)$.

These are established by writing down the long exact sequences of homologies of the three exact triangles in the diagram (3) and using the fact that $\mathcal{L}_\bullet, \mathcal{L}_\bullet^\#$ have only one nonzero homology. (*With a view to section 7 on formalism, we avoid local argument.*) By lemma 5.2, we can further assume that $\mathcal{R}_i = 0$ unless $(n - 1) + d \geq i \geq -(n - 1)$. Using induction there is a symmetric form $(\mathcal{Q}_\bullet, \omega)$ such that $x = [(\mathcal{E}_\bullet, \varphi)] = [(\mathcal{Q}_\bullet, \omega)]$, with $\mathcal{Q}_i = 0$ unless $d \geq i \geq 0$. By (4.4), \mathcal{Q}_\bullet is a resolution of $\mathcal{H}_0(\mathcal{Q}_\bullet)$. Further, ω induces a symmetric form $\omega_0 : \mathcal{H}_0(\mathcal{Q}_\bullet) \xrightarrow{\sim} \mathcal{H}_0(\mathcal{Q}_\bullet)^\vee$. By definition, $W(\zeta)([\mathcal{H}_0(\mathcal{Q}_\bullet, \omega_0)] = [(\mathcal{Q}_\bullet, \omega)] = x$. So, $W(\zeta)$ is surjective. The proof is complete. \square

The following is the main dévissage theorem for noetherian schemes.

Theorem 5.5. *Let X be a noetherian scheme, with $\dim X = d$, as in (2.1). Then, the homomorphism $W(\zeta) : W(\mathcal{A}(X)^\vee, \pm\tilde{\omega}) \longrightarrow W^d(D_{\mathcal{A}}^b(\mathcal{V}(X))_u^\pm)$ defined in (5.1), is an isomorphism.*

Proof. It follows from (5.4) that $W(\zeta)$ is surjective. The proof of injectivity is similar to that in the affine case ([MS1]), by an application of ([B3, 4.1]). We outline the proof briefly, in the case of +duality. Suppose (\mathcal{F}, φ_0) is a symmetric form in $(\mathcal{A}^\vee, \tilde{\omega})$ and $W(\zeta)([\mathcal{F}, \varphi_0]) = 0$. By definition 5.1 (see diagram 2) there is a finite resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$, with $\mathcal{E}_\bullet \in Ch^b(\mathcal{V}(X))$ and a quasi-isomorphism (without denominator) $\varphi : \mathcal{E}_\bullet \longrightarrow \mathcal{E}_\bullet^\#$, which lifts φ_0 and $W(\zeta)([\mathcal{F}, \varphi_0]) = [(\mathcal{E}_\bullet, \varphi)] = 0$. So, $(\mathcal{E}_\bullet, \varphi)$ is neutral in $D_{\mathcal{A}}^b(\mathcal{V}(X))$.

Going through the same argument in ([MS1, 5.12]) there is a hyperbolic from $(\mathcal{Q}, \varphi_1) \perp (\mathcal{Q}, -\varphi_1)$ such that $(\mathcal{E}, \varphi) \perp (\mathcal{Q}, \varphi_1) \perp (\mathcal{Q}, -\varphi_1)$ is neutral in $D_{\mathcal{A}}^b(\mathcal{V}(X))$. Therefore, it follows

$$(\mathcal{U}_{\bullet}, \beta) := \left(\mathcal{E}_{\bullet} \oplus \mathcal{Q}_{\bullet} \oplus \mathcal{Q}_{\bullet}^{\#}, \begin{pmatrix} \varphi_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \text{ is neutral in } (D_{\mathcal{A}}^b(\mathcal{V}(X))_u^{\pm}).$$

Therefore, $(\mathcal{U}_{\bullet}, \varphi)$ has a lagrangian $(\mathcal{L}_{\bullet}, \alpha)$ given by the following exact triangle

$$T^{-1}\mathcal{L}_{\bullet}^{\#} \xrightarrow{w} \mathcal{L}_{\bullet} \xrightarrow{\alpha} \mathcal{U}_{\bullet} \xrightarrow{\alpha^{\#}\varphi} \mathcal{L}_{\bullet}^{\#} \quad \text{with} \quad T^{-1}w^{\#} = w.$$

By the duality theorem 4.4, $\mathcal{H}_{-r}(\mathcal{U}_{\bullet}) \xrightarrow{\sim} \mathcal{H}_{d-r}(\mathcal{U}_{\bullet})^{\vee}$ and $\mathcal{H}_{-r}(\mathcal{L}_{\bullet}) \xrightarrow{\sim} \mathcal{H}_{d-r}(\mathcal{L}_{\bullet})^{\vee}$, for all $r \in \mathbb{Z}$. With these identification, the exact sequence of the homologies of the exact triangle reduces to

$$\begin{array}{ccccccc} \longrightarrow & \mathcal{H}_{-2}(\mathcal{L}_{\bullet})^{\vee} & \xrightarrow{\mathcal{H}_1(w)} & \mathcal{H}_1(\mathcal{L}_{\bullet}) & \xrightarrow{\mathcal{H}_1(\alpha)} & \mathcal{H}_1(\mathcal{U}_{\bullet}) & \xrightarrow{\mathcal{H}_{-1}(\alpha)^{\vee} \circ \mathcal{H}_1(\beta)} \mathcal{H}_{-1}(\mathcal{L}_{\bullet})^{\vee} \\ & & & & & & \\ \longrightarrow & \mathcal{H}_0(\mathcal{L}_{\bullet}) & \xrightarrow{\mathcal{H}_0(\alpha)} & \mathcal{H}_0(\mathcal{U}_{\bullet}) & \xrightarrow[\mathcal{H}_0(\alpha)^{\vee} \circ \mathcal{H}_0(\beta)]{} \mathcal{H}_0(\mathcal{L}_{\bullet})^{\vee} & \xrightarrow[\mathcal{H}_0(w)^{\vee}]{} \mathcal{H}_{-1}(\mathcal{L}_{\bullet})^{\vee\vee} & \\ & & & & & & \\ \longrightarrow & \mathcal{H}_1(\mathcal{U}_{\bullet})^{\vee} & \xrightarrow[\mathcal{H}_1(\alpha)^{\vee}]{} \mathcal{H}_1(\mathcal{L}_{\bullet})^{\vee} & \xrightarrow[\mathcal{H}_1(w)^{\vee}]{} \mathcal{H}_{-2}(\mathcal{L}_{\bullet})^{\vee\vee} & \xrightarrow[\mathcal{H}_2(\beta)^{\vee} \circ \mathcal{H}_{-2}(\alpha)^{\vee\vee}]{} \longrightarrow & & \end{array}$$

This exact sequence is "symmetric" and hence [B3, 4.1] applies. Since the sequence is exact, it follows $[(\mathcal{H}_0(\mathcal{U}_{\bullet}), \mathcal{H}_0(\varphi))] = [(0, 0)] = 0$ in $W(\mathcal{A}(X),^{\vee}, \tilde{\omega})$. However,

$$\begin{aligned} (\mathcal{H}_0(\mathcal{U}_{\bullet}), \mathcal{H}_0(\beta)) &= \left(\mathcal{H}_0(\mathcal{E}_{\bullet}) \oplus \mathcal{H}_0(\mathcal{Q}_{\bullet}) \oplus \mathcal{H}_0(\mathcal{Q}_{\bullet}^{\#}), \begin{pmatrix} \mathcal{H}_0(\varphi_0) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \\ &= (\mathcal{F}, \varphi_0) \perp \left(\mathcal{H}_0(\mathcal{Q}_{\bullet}) \oplus \mathcal{H}_0(\mathcal{Q}_{\bullet})^{\vee}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \end{aligned}$$

So, we have

$$[(\mathcal{F}, \varphi_0)] = \left[(\mathcal{F}, \varphi_0) \perp \left(\mathcal{H}_0(\mathcal{Q}_{\bullet}) \oplus \mathcal{H}_0(\mathcal{Q}_{\bullet})^{\vee}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right] = [(\mathcal{H}_0(\mathcal{U}_{\bullet}), \mathcal{H}_0(\varphi))] = 0.$$

The proof is complete. \square

6 The Final Results

So far we have been working with unsigned translation, in particular in the statement of theorem 5.5. To conform to the literature, in this sections we would present our results with respect to the standard signed translation. The readers are referred to ([B2], [MS1, §6]) for unexplained notations. We recall the following notations.

1. Denote $W_{St}^+(\mathcal{A}(X)) := W\left(\mathcal{A}(X)^\vee, (-1)^{\frac{d(d-1)}{2}} \tilde{\varpi}\right)$, and $W_{St}^-(\mathcal{A}(X)) := W\left(\mathcal{A}(X)^\vee, -(-1)^{\frac{d(d-1)}{2}} \tilde{\varpi}\right)$.
2. Now on, $T := T_s : D^b(\mathcal{V}(X)) \longrightarrow D^b(\mathcal{V}(X))$ will denote the signed ("standard") translation. For $n \in \mathbb{Z}$, denote $\zeta_n = T^{-n} \circ \zeta : \mathcal{A}(X) \longrightarrow D^b(\mathcal{V}(X))$, where ζ is as in (3.10). Note $\zeta_0 = \zeta$.

Now, we state our results.

Theorem 6.1. *Suppose X is a noetherian scheme, with $\dim X = d$, as in (2.1). Then,*

1. *The functor $\zeta_0 : \mathcal{A}(X) \longrightarrow D_{\mathcal{A}(X)}^b(\mathcal{V}(X))$ induces an isomorphism*

$$W(\zeta_0) : W_{St}^+(\mathcal{A}(X)) \xrightarrow{\sim} W^d(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, \varpi).$$

2. *The functor $\zeta_1 : \mathcal{A}(X) \longrightarrow D_{\mathcal{A}(X)}^b(\mathcal{V}(X))$ induces an isomorphism*

$$W_{St}^-(\mathcal{A}(X)) \xrightarrow{\sim} W^{d-2}(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, -\varpi).$$

3. *For $n = d - 1, d - 3$, we have $W^n(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, \pm \varpi) = 0$.*

*Further, 4-periodicity determines all the shifted Witt groups $W^n(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, \pm \varpi)$.*

Proof. Follows from theorem 5.5, as in ([MS1, §6]). □

Remark. Following ([MS1]), in the case of Cohen-Macaulay affine schemes $X = \text{Spec}(A)$, a version of (6.1) was given by Sanders and Sane for resolving subcategories of $\text{Mod}(A)$ associated to semi-dualizing modules.

We would also consider the Witt groups of the derived category $D^b(\mathcal{A}(X))$ and of its full subcategory $D_{\mathcal{A}}^b(\mathcal{A}(X))$. As was the case with $D_{\mathcal{A}(X)}^b(\mathcal{V}(X))$, $D_{\mathcal{A}(X)}^b(\mathcal{A}(X))$ may not have a triangulated structure. The following follows from the formalism given in ([MS1]).

Theorem 6.2. *Suppose X a noetherian scheme, as in (2.1), with $\dim X = d$. The duality ${}^\vee : \mathcal{A} \longrightarrow \mathcal{A}$ induces a duality on the derived category $D^b(\mathcal{A})$, which we continue to denote by ${}^\vee$. Then, $D_{\mathcal{A}}^b(\mathcal{A})$ is stable under this duality. Further, the functors $\mathcal{A} \longrightarrow D_{\mathcal{A}}^b(\mathcal{A}) \hookrightarrow D^b(\mathcal{A})$ induce the following triangle of isomorphisms*

$$\begin{array}{ccc} W(\mathcal{A}^\vee, \pm \tilde{\varpi}) & \xrightarrow{\sim} & W(D_{\mathcal{A}}^b(\mathcal{A}^\vee, \pm \tilde{\varpi})) \\ & \searrow \sim & \downarrow \wr \\ & & W(D^b(\mathcal{A}^\vee, \pm \tilde{\varpi})) \end{array}$$

Proof. Since \mathcal{A} has 2-out-of-3 property, the theorem follows from [MS1, A.1, A.2]. \square

The following decomposition of the derived Witt groups is line with the regular case.

Theorem 6.3. *Suppose X is a noetherian scheme, with $\dim X = d$, as in (2.1) and $X^{(d)}$ will denote the set of all closed points of codimension n in X . Then, the homomorphisms*

$$W^d(D_{\mathcal{A}}^b(\mathcal{V}(X)), *, 1, \varpi) \xrightarrow{\sim} \bigoplus_{x \in X^{(d)}} W^d(D_{\mathcal{A}(\mathcal{O}_{X,x})}^b(\mathcal{O}_{X,x}), *, 1, \varpi) \quad \text{and}$$

$$W^{d-2}(D_{\mathcal{A}}^b(\mathcal{V}(X)), *, 1, -\varpi) \xrightarrow{\sim} \bigoplus_{x \in X^{(d)}} W^{d-2}(D_{\mathcal{A}(\mathcal{O}_{X,x})}^b(\mathcal{O}_{X,x}), *, 1, -\varpi)$$

are isomorphisms.

Proof. We prove only the first isomorphism. The diagram:

$$\begin{array}{ccc} W_{St}(\mathcal{A}(X))^+ & \xrightarrow[\sim]{W(\zeta)} & W^d(D_{\mathcal{A}(X)}^b(\mathcal{V}(X)), *, 1, \varpi) \\ \eta \downarrow & & \downarrow \gamma \\ \bigoplus_{x \in X^{(d)}} W_{St}(\mathcal{A}(\mathcal{O}_{X,x}))^+ & \xrightarrow[\oplus W(\zeta)]{} & \bigoplus_{x \in X^{(d)}} W^d(D_{\mathcal{A}(\mathcal{O}_{X,x})}^b(\mathcal{O}_{X,x}), *, 1, \varpi) \end{array}$$

By theorem (6.1), two horizontal homomorphisms are isomorphisms. Also note, $\mathcal{A}(X) \rightarrow \coprod_{x \in X^{(d)}} \mathcal{A}(\mathcal{O}_{X,x})$ is a duality preserving equivalence of categories (*in affine case, this is the Chinese remainder theorem*). Therefore, η is an isomorphism. Now, since three other maps in this rectangle are isomorphisms, so is γ . The proof is complete. \square

Remark 6.4. When X is regular, (6.3) is a result of Balmer and Walter ([BW]). In this case, (6.3) would follow from the equivalence of the corresponding categories $D_{\mathcal{A}}^b(X) \rightarrow \coprod_{x \in X^{(d)}} \mathcal{A}(\mathcal{O}_{X,x})$ ([BW, 7.1]).

Remark 6.5. It was implicitly assumed in the statement of theorem 6.1 and others, that $\mathcal{A}(X)$ has nonzero objects. This would be false, if $\text{depth}(\mathcal{O}_{X,x}) < d = \dim X$ for all closed points x . If such cases, a version of these results can be given by replacing d by $\delta := \max\{\text{depth}(\mathcal{O}_{X,x}) : x \text{ is a closed point of } X\}$ (see § 7).

7 Witt-Dévissage formalism

Some authors considered resolving subcategories of the category $\text{Mod}(A)$ of modules over commutative noetherian rings A . Resolving subcategories of abelian categories was defined (see 3.11), by substituting the requirement that A is in resolving subcategory, by the

condition (3) of the definition (3.11). In this section, we give formal versions of results in § 6, for resolving subcategories (see 3.11) of abelian categories. There is a list of examples of resolving subcategories of $Mod(A)$ in ([T1]). The one that is of our particular interest is the one corresponding to the notion of semi-dualizing A -modules ω . Such semi-dualizing A -modules ω naturally give rise to resolving subcategories of $Mod(A)$. With that in mind, we define the following and establish our set up for this section.

Definition 7.1. Suppose \mathcal{V} is a resolving subcategory of an abelian category \mathcal{C} . Let ω be a fixed object on \mathcal{C} and write $M^* := Mor(M, \omega)$. Assume that ω has an injective resolution, and choose one such resolution \mathcal{I}_\bullet , as follows:

$$0 \longrightarrow \omega \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}_2 \longrightarrow \dots \quad (4)$$

1. Given any object M in \mathcal{C} , define $Ext^i(M, \omega) := H^i(M, \mathcal{I}_\bullet)$. (*We only define $Ext^i(-, \omega)$; and not giving any definition of $Ext^i(-, \mathcal{G})$ when $\mathcal{G} \neq \omega$.*) It immediately follows that $\{M \mapsto Ext^i(-, \omega) : i \geq 0\}$ is a contravariant δ -functor (see [H, §III.1]).
2. We say that \mathcal{V} inherits a ω -**duality structure** if, for all objects P in \mathcal{V} , (1) $P^* \in \mathcal{V}$, (2) and there is a natural equivalence $ev : P \xrightarrow{\sim} P^{**}$, from the identity functor $1_{\mathcal{V}}$ to the double dual.
3. Also, we say that \mathcal{V} is **totally ω -reflexive** if in addition, for all objects P in \mathcal{V} and integers $i \geq 1$, $Ext^i(P, \omega) = 0$. It is easy to see that if $P_\bullet \rightarrow M$ is a \mathcal{V} -resolution of an object in \mathcal{C} that there are natural isomorphisms from the homologies $H_i(P_\bullet^\#) \xrightarrow{\sim} Ext^i(M, \omega)$, where $P_\bullet^\#$ denote the dual of P_\bullet induced by $*$.
4. We also recall some standard definitions. For objects M in \mathcal{C} , $\dim_{\mathcal{V}} M$ will denote the length of the shortest \mathcal{V} -resolution of M (which can be infinite). As in § 3.1, $\mathcal{B} := \mathcal{B}(\mathcal{V}) := \{M \in \mathcal{C} : \dim_{\mathcal{V}} M < \infty\}$ denotes the full subcategory of such objects. Also denote, $d := \dim_{\mathcal{V}}(\mathcal{B}) := \max\{\dim_{\mathcal{V}} M : M \in \mathcal{B}\}$. If $d = \dim_{\mathcal{V}}(\mathcal{B}) < \infty$, let

$$\mathcal{A}(\omega) := \mathcal{A}(\mathcal{V}, \omega) := \{M \in \mathcal{B}(\mathcal{V}) : Ext^i(M, \omega) = 0 \ \forall i < d.\}$$

denote the full subcategory. It follows $\mathcal{B}(\mathcal{V})$ and $\mathcal{A}(\omega)$ are exact subcategories of \mathcal{C} .

5. **Set up:** In what follows, we will have the following set up:

\mathcal{C} will denote an abelian category and \mathcal{V} will be a resolving subcategory of \mathcal{C} . We fix an object ω with a (chosen) injective resolution $\omega \hookrightarrow \mathcal{I}_\bullet$. The δ -functor $\{Ext^i(-, \omega)\}$ is defined as above. We assume: (1) \mathcal{V} inherits a ω -duality structure and is totally ω -reflexive. (2) Given an object $\mathcal{F} \in \mathcal{B}$ and a \mathcal{V} -resolution $\mathcal{E}_\bullet \rightarrow \mathcal{F}$, the cycle objects $Z_n := \ker(\mathcal{E}_n \rightarrow \mathcal{E}_{n-1}) \in \mathcal{V} \ \forall n \gg 0$. (3) Further, $d := \dim_{\mathcal{V}}(\mathcal{B}) < \infty$. We observed (3.13), under this set up

- (a) $\mathcal{B}(\mathcal{V})$ is an exact subcategory and every epimorphism in \mathcal{B} is admissible.
- (b) Further, the functor $\zeta : \mathcal{B}(\mathcal{V}) \longrightarrow D^b(\mathcal{V})$ is defined (see 3.6).
- (c) It also follows that $\mathcal{A}(\omega)$ is an exact subcategory of \mathcal{C} .

Most of what are in § 6, work for resolving subcategories \mathcal{V} , of abelian categories \mathcal{C} , as in the set up (5) of 7.1. We state them below.

Lemma 7.2. *Suppose $(\mathcal{C}, \mathcal{V}, \omega)$ is as in (5) of 7.1. Then, the association $M \mapsto M^\vee := \text{Ext}^d(M, \omega)$ defines a duality ${}^\vee : \mathcal{A}(\omega) \longrightarrow \mathcal{A}(\omega)$.*

Proof. Note, $M \in \mathcal{A}(\omega) \implies M^\vee \in \mathcal{A}(\omega)$. The rest of the proof is as that of (4.2). \square

Theorem 7.3. *Suppose $(\mathcal{C}, \mathcal{V}, \omega)$ be as in (5) of 7.1. Suppose \mathcal{E}_\bullet is a complex in $Ch_{\mathcal{A}}^b(\mathcal{V})$. Then, the dual $\mathcal{E}_\bullet^\#$ is also $Ch_{\mathcal{A}}^b(\mathcal{V})$. Further, there is a canonical isomorphism $\eta : H_{-r}(\mathcal{E}_\bullet^\#) \xrightarrow{\sim} H_{r-d}(\mathcal{E}_\bullet)^\vee$, which is natural with respect to morphisms in $Ch_{\mathcal{A}}^b(\mathcal{V})$.*

Proof. Same as that of (4.4). \square

Theorem 7.4. *Suppose $(\mathcal{C}, \mathcal{V}, \omega)$ be as in (5) of 7.1 and the rest of the notations be same as in § 6. Also refer to the definition (3.13) of the functor $\zeta : \mathcal{B}(\mathcal{V}) \longrightarrow D^b(\mathcal{V})$. Then,*

1. *The functor $\zeta_0 : \mathcal{A}(\omega) \longrightarrow D_{\mathcal{A}(\omega)}^b(\mathcal{V})$ induces an isomorphism*

$$W(\zeta_0) : W_{St}^+(\mathcal{A}(\omega)) \xrightarrow{\sim} W^d(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, \varpi).$$

2. *The functor $\zeta_1 : \mathcal{A}(\omega) \longrightarrow D_{\mathcal{A}(\omega)}^b(\mathcal{V})$ induces an isomorphism*

$$W_{St}^-(\mathcal{A}(\omega)) \xrightarrow{\sim} W^{d-2}(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, -\varpi).$$

3. *For $n = d - 1, d - 3$, we have $W^n(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, \pm\varpi) = 0$.*

*Further, 4-periodicity determines all the shifted Witt groups $W^n(D_{\mathcal{A}(\omega)}^b(\mathcal{V}), *, 1, \pm\varpi)$.*

Proof. Similar to that of theorem 6.1. \square

Theorem 7.5. *Suppose $(\mathcal{C}, \mathcal{V}, \omega)$ is as in (5) of 7.1. Then, $D_{\mathcal{A}(\omega)}^b(\mathcal{A}(\omega))$ is stable under the duality on $D^b(\mathcal{A}(\omega))$ induced by ${}^\vee : \mathcal{A}(\omega) \rightarrow \mathcal{A}(\omega)$ (which we continue to denote by ${}^\vee$). Further, the functors $\mathcal{A}(\omega) \rightarrow D_{\mathcal{A}(\omega)}^b(\mathcal{A}(\omega)) \hookrightarrow D^b(\mathcal{A}(\omega))$ induce the following triangle of isomorphisms*

$$\begin{array}{ccc} W(\mathcal{A}(\omega), {}^\vee, \pm \tilde{\omega}) & \xrightarrow{\sim} & W(D_{\mathcal{A}(\omega)}^b(\mathcal{A}(\omega), {}^\vee, \pm \tilde{\omega})) \\ & \searrow \sim & \downarrow \wr \\ & & W(D^b(\mathcal{A}(\omega), {}^\vee, \pm \tilde{\omega})) \end{array}$$

Proof. Note that the diagonal isomorphism follows directly from ([B3, 4.7]). So, we need only to prove that the horizontal map is surjective. The proof would be very similar to that of (6.2), by an application [MS1, A.1, A.2]. Some clarification is needed, because $\mathcal{A}(\omega)$ does not have the 2-out-of-3 property, which was used in (6.2). First, we claim that for complexes $(\mathcal{E}_\bullet, \partial_\bullet)$ in $Ch_{\mathcal{A}(\omega)}^b(\mathcal{A}(\omega))$ all the boundaries $B_i := \text{image}(\partial_{i-1}) \subseteq \mathcal{E}_i$ and cycles $Z_i := \ker(\partial_i) \subseteq \mathcal{E}_i$ are in $\mathcal{A}(\omega)$. To see this note, since $\mathcal{B}(\mathcal{V})$ is exact and epimorphisms in $\mathcal{B}(\mathcal{V})$ are admissible, inducting from right, it follows $B_i, Z_i, \frac{\mathcal{E}_i}{B_i}$ are in $\mathcal{B}(\mathcal{V})$. Now, for $X = B_i, Z_i, \frac{\mathcal{E}_i}{B_i}$, by induction from left, it follows $\text{Ext}^i(X, \omega) = 0$ for all $i \neq d$, hence are in $\mathcal{A}(\omega)$. This establishes the claim. In fact, proof of [MS1, A.1, A.2] works whenever, $B_i, Z_i, \frac{\mathcal{E}_i}{B_i}$ are in $\mathcal{A}(\omega)$ for all i . This completes the proof.

We comment that one can combine 3.13 and 7.3 to give a proof of the duality statement, by constructing a double complex in $D^b(\mathcal{V})$, as in ([MS1, §3]). \square

References

- [AB] Auslander, Maurice; Bridger, Mark *Stable module theory*. Memoirs of the AMS, No. 94 AMS, Providence, R.I. 1969 146 pp.
- [B1] Balmer, Paul *Derived Witt groups of a scheme*. J. Pure Appl. Algebra 141 (1999), no. 2, 101-129.
- [B2] Balmer, Paul *Triangular Witt Groups Part I: The 12-Term Localization Exact Sequence*, K-Theory 19: 311-63, 2000
- [B3] Balmer, Paul *Triangular Witt groups. II. From usual to derived*. Math. Z. 236 (2001), no. 2, 351-382.
- [BW] Balmer, Paul; Walter, Charles A *Gersten-Witt spectral sequence for regular schemes*. Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 1, 127-152

- [BIK] Benson, David J.; Iyengar, Srikanth B.; Krause, Henning *Stratifying modular representations of finite groups*. Ann. of Math. (2) 174 (2011), no. 3, 1643-1684.
- [H] Hartshorne, Robin *Algebraic geometry*. GTM, No. 52. 1977. xvi+496 pp
- [MS1] Satya Mandal and Sarang Sane, *On Dévissage for Witt groups*, Preprint (see Arxiv)
- [MS2] Satya Mandal and Sarang Sane, *Stability of locally CMFPD homologies under duality*, Preprint
- [MS3] Satya Mandal and Sarang Sane, *Filtered Witt Groups*, Preprint
- [T1] Takahashi, Ryo *Classifying thick subcategories of the stable category of Cohen-Macaulay modules*. Adv. Math. 225 (2010), no. 4, 2076-2116.
- [T2] Takahashi, Ryo *Modules in resolving subcategories which are free on the punctured spectrum*. Pacific J. Math. 241 (2009), no. 2, 347-367.
- [W] Weibel, Charles A. it An introduction to homological algebra. Cambridge Stud. in Adv. Math, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp.